

# LITTLEWOOD-RICHARDSON COEFFICIENTS FOR REFLECTION GROUPS

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ABSTRACT. In this paper we explicitly compute all Littlewood-Richardson coefficients for semisimple and Kac-Moody groups  $G$ , that is, the structure constants (also known as the *Schubert structure constants*) of the cohomology algebra  $H^*(G/P, \mathbb{C})$ , where  $P$  is a parabolic subgroup of  $G$ . These coefficients are of importance in enumerative geometry, algebraic combinatorics and representation theory. Our formula for the Littlewood-Richardson coefficients is purely combinatorial and is given in terms of the Cartan matrix and the Weyl group of  $G$ . However, if some off-diagonal entries of the Cartan matrix are 0 or  $-1$ , the formula may contain negative summands. On the other hand, if the Cartan matrix satisfies  $a_{ij}a_{ji} \geq 4$  for all  $i, j$ , then each summand in our formula is nonnegative that implies nonnegativity of all Littlewood-Richardson coefficients. We extend this and other results to the structure coefficients of the  $T$ -equivariant cohomology of flag varieties  $G/P$  and Bott-Samelson varieties  $\Gamma_1(G)$ .

## CONTENTS

1. Introduction	1
2. Definitions and main results	3
3. Twisted group algebras and generalized Littlewood-Richardson coefficients	11
4. Generalized nil Hecke algebras and proof of Theorem 2.14	25
5. Proof of Theorems 2.10, 2.4 and 2.13	33
5.1. Proof of Theorem 2.10	33
5.2. Proof of Theorems 2.4 and 2.13	35
6. Positivity of Littlewood-Richardson coefficients and proof of Theorem 2.16	39
7. Examples	42
7.1. The rank 2 case	42
7.2. Finite Type $A$ examples	46
7.3. Finite type $H_3$ examples	48
References	49

## 1. INTRODUCTION

The goal of this paper is to explicitly compute the Littlewood-Richardson coefficients which are structure constants of the cohomology algebra  $H^*(G/B, \mathbb{C})$  for the

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flag variety  $G/B$  of an arbitrary semisimple or Kac-Moody group. More precisely, let  $W$  be the Weyl group of  $G$  and let  $\sigma_w \in H^*(G/B, \mathbb{C})$  denote the Schubert cocycle corresponding to an element  $w \in W$ . Then the Littlewood-Richardson coefficients  $c_{u,v}^w \in \mathbb{Z}_{\geq 0}$ ,  $u, v, w \in W$  are defined as the structure constants of the cup product in  $H^*(G/B, \mathbb{C})$  with respect to the basis  $\{\sigma_w, w \in W\}$ :

$$(1.1) \quad \sigma_u \cup \sigma_v = \sum_{w \in W} c_{u,v}^w \sigma_w .$$

The study of Littlewood-Richardson coefficients  $c_{u,v}^w$  has a long history and is an essential part of Schubert calculus. In enumerative geometry, these coefficients are realized as the cardinality of triple intersections of certain Schubert varieties in general position. From this point of view, there have been several formulas for  $c_{u,v}^w$  in using transversality and degeneration techniques [2, 6, 11, 25, 45, 49, 48, 54]. In algebraic combinatorics, the numbers  $c_{u,v}^w$  for special  $u, v, w$  can be determined via puzzles or counting problems using Young tableaux [14, 15, 26, 32, 39, 51]. Other combinatorial approaches for computing  $c_{u,v}^w$  include coinvariant algebras with Schubert polynomial bases ([5, 9, 13, 36, 44]) or recursions over the Weyl groups ([8, 31]). While there have been many interesting formulas and algorithms for computing these numbers, they are mostly limited to special cases of reductive Lie groups  $G$ .

To the best of our knowledge, the only non-recursive formula for Littlewood-Richardson coefficients was obtained by H. Duan in a remarkable paper [12] and the equivariant generalization was later obtained by M. Willems in [57]. The major issue here is that both Duan's and Willems formulas contain a large number of summands, including several negative terms. For instance, if  $G = \widehat{SL}_2$ ,  $u = v = (s_1 s_2)^2$ ,  $w = u^2$ , then Duan's formula for  $c_{u,v}^w$  has about 17,000 summands, but our formula (2.3) contains only 19 summands (see Remark 2.8 for details). However, a comparison with Duan's and Willems approaches was very productive and resulted in a discovery of a combinatorial formula for *Bott-Samelson* numbers, i.e., structure constants of the (equivariant) cohomology algebra of Bott-Samelson varieties (Theorem 2.10).

**Remark 1.1.** In fact, the Littlewood-Richardson coefficients for partial flag varieties  $G/P$ , where  $P \supset B$  (e.g., for Grassmannians) is a parabolic subgroup of  $G$  are determined by the respective coefficients for  $G/B$  because the pullback  $H^*(G/P, \mathbb{C}) \hookrightarrow H^*(G/B, \mathbb{C})$  of canonical projection  $G/B \rightarrow G/P$  turns  $H^*(G/P, \mathbb{C})$  into the subalgebra of  $H^*(G/B, \mathbb{C})$  spanned by a part of the Schubert basis.

**Remark 1.2.** In some of the above mentioned papers and several other papers the coefficients  $c_{u,v}^w$  have been referred to as *Schubert structure constants*. However, we believe that the ‘‘Littlewood-Richardson’’ terminology for these constants is justified historically (it is used e.g., in [5, 7, 10, 11, 19, 26, 28, 29, 30, 38, 41, 42, 43, 44, 54]) and by a number of purely mathematical reasons. First, the classical Jacobi-Trudy formula for Schubert classes in Grassmannians  $Gr_k(\mathbb{C}^n)$  (see e.g., [10]) implies that the structure constants of  $H^*(Gr_k(\mathbb{C}^n), \mathbb{C})$  are the classical Littlewood-Richardson coefficients (in fact, [10, Theorem 2] asserts that the structure constants for maximal isotropic Grassmannians are also given by Littlewood-Richardson-Stembridge rule).

Second, based on results of Klyachko on Horn inequalities and follow-up work (see e.g., [2, 4, 22, 27, 46]), if  $V_\lambda, V_\mu, V_\nu$  are simple  $G$ -modules and  $c'_{\lambda,\mu}$  denotes the multiplicity  $\dim_{\mathbb{C}} \operatorname{Hom}_G(V_\nu, V_\lambda \otimes V_\mu)$ , then the set of triples  $(\lambda, \mu, \nu)$  such that  $c'_{\lambda,\mu} \neq 0$  is determined (up to saturation) in terms of the set of all triples  $(u, v, w) \in W^3$  such that  $c'_{u,v} \neq 0$ . Another similarity (and complementarity) of these coefficients is that, when  $\dim G < \infty$ , the representation ring  $R(G)$  (the carrier of the “genuine” Littlewood-Richardson coefficients  $c'_{\lambda,\mu}$ ) is the invariant algebra  $\mathbb{C}[T]^W$ , where  $T$  is the maximal torus of  $G$  and the cohomology algebra  $H^*(G/B, \mathbb{C})$  is the coinvariant algebra  $\mathbb{C}[\operatorname{Lie}(T)]_W$  of the Lie algebra  $\operatorname{Lie}(T)$ .

We compute  $c'_{u,v}$  in terms of the  $W$ -action on the root lattice of  $G$ . Our main formula (2.3) is very different from Duan’s and, in particular, if the Cartan matrix satisfies  $a_{ij}a_{ji} \geq 4$  for all  $i, j$ , then all summands in (2.3) turn out to be nonnegative which implies  $c'_{u,v} \geq 0$  for all relevant  $u, v, w$  (Theorem 2.16). However, if the Cartan matrix  $A$  of  $G$  has entries  $a_{ij} \in \{0, -1\}$ , the right hand side of (2.3) may contain negative summands. Nevertheless, we believe that the negative terms can be effectively canceled (see Remark 2.7 for details). We discuss positivity in greater details in Theorem 2.16 and Conjectures 2.18 and 2.19.

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## 2. DEFINITIONS AND MAIN RESULTS

Let  $G$  be a Kac-Moody group and let  $A = (a_{ij})$  be the  $I \times I$  Cartan matrix of  $G$  (where  $I$  denotes the indexing set, i.e., the set of vertices of the Dynkin diagram). The Weyl group  $W$  of  $G$  is generated by simple reflections  $s_i$ ,  $i \in I$  that act on the root space  $V = \bigoplus_{i \in I} \mathbb{C} \cdot \alpha_i$  by:

$$(2.1) \quad s_i(\alpha_j) = \alpha_j - a_{ij}\alpha_i$$

for  $i, j \in I$ .

**Definition 2.1.** Let  $m$  be a positive integer and let  $\mathbf{i} \in I^m$ . For each subset  $M = \{m_1 < \dots < m_r\}$  of the interval  $[m] := \{1, 2, \dots, m\}$  denote by  $\mathbf{i}_M$  the subsequence  $(i_{m_1}, \dots, i_{m_r}) \in I^r$  of  $\mathbf{i}$ . We say that a sequence  $\mathbf{i} = (i_1, \dots, i_m) \in I^m$  is *reduced* if the element  $w = w_{\mathbf{i}} := s_{i_1} \dots s_{i_m} \in W$  is shortest possible and define its *Coxeter length*  $\ell(w) := m$ . We say that a sequence  $\mathbf{i}$  is *admissible* if  $i_k \neq i_{k+1}$  for all  $j \in [m-1]$  (clearly, every reduced sequence is admissible). Given  $w \in W$ , denote by  $R(w)$  the set of all *reduced words* of  $w$ , i.e., all  $\mathbf{i} \in I^{\ell(w)}$  such that  $w_{\mathbf{i}} = w$ .

**Definition 2.2.** Let  $m \geq 0$  and let  $L, M$  be subsets of  $[m]$  such that  $|L| + |M| = m$ . We say that a bijection

$$\varphi : L \xrightarrow{\sim} [m] \setminus M$$

is *bounded* if  $\varphi(\ell) < \ell$  for each  $\ell \in L$ . Given a sequence  $\mathbf{i} = (i_1, \dots, i_m) \in I^m$ , we say that a bounded bijection  $\varphi : L \xrightarrow{\sim} [m] \setminus M$  is  *$\mathbf{i}$ -admissible* if the sequence  $\mathbf{i}_{M \cup \varphi(L_{<\ell})}$  is admissible for all  $\ell \in L$ , where we abbreviate  $L_{<\ell} := L \cap [\ell - 1]$  (in particular  $L_{<\ell} = \emptyset$  if  $\ell$  is the minimal element of  $L$ ).

**Example 2.3.** Let  $I = \{1, 2\}$  with  $\mathbf{i} = (1, 2, 1, 2)$ . If  $L = M = \{3, 4\}$ , then

$$\phi_1 : (3, 4) \rightarrow (2, 1) \quad \phi_2 : (3, 4) \rightarrow (1, 2).$$

are both bounded bijections. In particular,  $\phi_1$  is  $\mathbf{i}$ -admissible and  $\phi_2$  is not  $\mathbf{i}$ -admissible.

For  $j \in I$  denote by  $\langle \cdot, \alpha_j^\vee \rangle$  the linear function on  $V$  given by  $\langle \alpha_i, \alpha_j^\vee \rangle = a_{ij}$ . For any sequence  $\mathbf{i} = (i_1, \dots, i_m) \in I^m$  and bijection  $\varphi : L \xrightarrow{\sim} [m] \setminus M$  we define the integer  $p_\varphi$  by the formula

$$(2.2) \quad p_\varphi := \prod_{\ell \in L} \langle w_{\mathbf{i}_{M(\ell)}}(-\alpha_{i_\ell}), \alpha_{i_{\varphi(\ell)}}^\vee \rangle$$

where

$$M(\ell) := \{r \in M \cup \varphi(L_{<\ell}) \mid \varphi(\ell) < r < \ell\}.$$

If  $M(\ell) = \emptyset$ , then  $w_{\mathbf{i}_{M(\ell)}} = 1$  and if  $L = \emptyset$  then  $p_\varphi = 1$ . The following is our main result (in which we implicitly use the well-known fact that  $c_{u,v}^w = 0$  unless  $\ell(w) = \ell(u) + \ell(v)$ ).

**Theorem 2.4.** *Let  $G$  be a Kac-Moody group and  $W = \langle s_i, i \in I \rangle$  be its Weyl group. Then for any  $u, v, w \in W$  such that  $\ell(w) = \ell(u) + \ell(v)$  and any given  $\mathbf{i} \in R(w)$  one has:*

$$(2.3) \quad c_{u,v}^w = \sum p_\varphi$$

with the summation over all triples  $(\mathbf{u}, \mathbf{v}, \varphi)$ , where

- $\mathbf{u}, \mathbf{v} \subset [m]$  such that  $\mathbf{i}_{\mathbf{u}} \in R(u)$ ,  $\mathbf{i}_{\mathbf{v}} \in R(v)$  (hence  $|\mathbf{u} \cap \mathbf{v}| + |\mathbf{u} \cup \mathbf{v}| = m$ );
- $\varphi : \mathbf{u} \cap \mathbf{v} \rightarrow [m] \setminus (\mathbf{u} \cup \mathbf{v})$  is an  $\mathbf{i}$ -admissible bounded bijection.

**Remark 2.5.** The right hand side of (2.3) depends on a choice of  $\mathbf{i} = (i_1, \dots, i_m) \in R(w)$ . It would be interesting to find an “optimal”  $\mathbf{i}$  (depending on  $u, v$  and  $w$ ) that would minimize the number of summands.

**Remark 2.6.** It turns out that (2.3) holds if we drop the condition of  $\mathbf{i}$ -admissibility. See discussion after Theorem 2.10 for details.

**Remark 2.7.** It frequently happens that each summand of (2.3) is positive (see Theorem 2.16 for details). Based on this observation, we can define for each  $u, v, w \in W$ , and  $\mathbf{i} \in R(w)$  the coefficient  $c_{u,v}^{\mathbf{i},+}$  by replacing each  $p_\varphi$  in (2.3) with  $p_\varphi^+$ , where  $p_\varphi^+$  is obtained by replacing each factor  $p_\ell := \langle w_{\mathbf{i}_{M(\ell)}}(-\alpha_{i_\ell}), \alpha_{i_{\varphi(\ell)}}^\vee \rangle$  in (2.2) with  $\max(p_\ell, 0)$ . We then define

$$c_{u,v}^{w,+} := \min_{\mathbf{i} \in R(w)} c_{u,v}^{\mathbf{i},+}.$$

Based on numerous examples, including all  $G = SL_n$ ,  $n \leq 6$ , we can conjecture that

$$(2.4) \quad c_{u,v}^w \leq c_{u,v}^{w,+}$$

for all  $u, v, w \in W$  with  $\ell(u) + \ell(v) = \ell(w)$ . In fact, in most of examples, the inequality (2.4) was an equality. In any case, if the inequality (2.4) holds, we can try to express  $c_{u,v}^w$  as a “sub-sum” of one of  $c_{u,v}^{\mathbf{i},+} = \sum p_{\varphi}^+$ , i.e., express the Littlewood-Richardson coefficient  $c_{u,v}^w$  in purely nonnegative terms.

**Remark 2.8.** H. Duan proved in [12] that if  $G$  is semisimple, then for each  $u, v, w \in W$  such that  $\ell(w) = \ell(u) + \ell(v)$  and a given  $\mathbf{i} = (i_1, \dots, i_m) \in R(w)$  one has in the notation as above:

$$(2.5) \quad c_{u,v}^w = \sum \prod_{1 < \ell \leq m} \frac{c_{*,\ell}!}{\prod_{1 \leq k < \ell} c_{k,\ell}!} \prod_{1 \leq k < \ell \leq m} b_{k,\ell}^{c_{k,\ell}}$$

where  $b_{k,\ell} = \langle s_{i_{k+1}} \cdots s_{i_{\ell-1}}(-\alpha_{i_{\ell}}), \alpha_{i_k}^{\vee} \rangle$ ,  $1 \leq k < \ell \leq m$  and the summation is over all triples  $(\mathbf{u}, \mathbf{v}, \mathbf{c})$  such that

- $\mathbf{u}, \mathbf{v} \subset [1, m]$ ,  $\mathbf{i}_{\mathbf{u}} \in R(w)$ ,  $\mathbf{i}_{\mathbf{v}} \in R(v)$ .
- $\mathbf{c} = (c_{k,\ell} | 1 \leq k < \ell \leq m)$  is a triangular array of nonnegative integers such that

$$(2.6) \quad c_{*,k} - c_{k,*} = \begin{cases} 1 & \text{if } k \in \mathbf{u} \cap \mathbf{v} \\ -1 & \text{if } k \in \mathbf{u} \cup \mathbf{v} \setminus (\mathbf{u} \cap \mathbf{v}) \\ 0 & \text{otherwise} \end{cases}$$

for  $k = 1, \dots, m$  (here we abbreviated  $c_{*,k} = \sum_{1 \leq s < k} c_{s,k}$ ,  $c_{k,*} = \sum_{k < s \leq m} c_{k,s}$ ).

Even though Duan’s formula (2.5) makes sense for any Kac-Moody group  $G$  and bears some resemblance with our formula (2.3) (see also Remark 2.12), the formulas are still very different: the set of all triangular arrays  $\mathbf{c}$  in (2.5) is much larger than the set of all relevant bounded bijections in Theorem 2.4. For instance, if  $G = \widehat{SL}_2$ ,  $u = v = (s_1 s_2)^2$ ,  $w = u^2$ , then Duan’s formula for  $c_{u,v}^w$  has about 17,000 summands, but our formula (2.3) contains only 19 summands.

In fact, Theorem 2.4 is a particular case of more general result (Theorem 2.13) in which we compute all  $T$ -equivariant Littlewood-Richardson coefficients  $p_{u,v}^w \in S^{\ell(u)+\ell(v)-\ell(w)}(V)$  that are defined by:

$$(2.7) \quad \sigma_u^T \cup \sigma_v^T = \sum_{w \in W} p_{u,v}^w \sigma_w^T,$$

where  $H_T^*(G/B)$  is the  $T$ -equivariant cohomology algebra of  $G/B$  (see e.g., [34, Section 11.3]),  $T \subset B$  is a maximal torus of  $G$ , and  $\sigma_w^T$  is the  $T$ -equivariant Schubert cocycle (it is well-known that  $p_{u,v}^w$  is homogeneous of degree  $\ell(u) + \ell(v) - \ell(w)$ , in particular,  $c_{u,v}^w = \delta_{\ell(w), \ell(u)+\ell(v)} p_{u,v}^w$ ).

Using results of [12, 56, 57], we compute  $p_{u,v}^w$  via the *Bott-Samelson coefficients*  $p_{K',K''}^{\mathbf{i},K}$  as follows.

Recall that for each sequence  $\mathbf{i} = (i_1, \dots, i_m) \in I^m$  the  $\mathbf{i}$ -th Bott-Samelson variety  $\Gamma_{\mathbf{i}} = \Gamma_{\mathbf{i}}(G)$  of  $G$  is defined by:

$$\Gamma_{\mathbf{i}} = (P_{i_1} \times_B P_{i_2} \times_B \cdots \times_B P_{i_m})/B$$

where  $P_i$ ,  $i \in I$  stands for the  $i$ -th minimal parabolic subgroup (see e.g., [56]).

It is well-known (see e.g., [1, 34]) that the  $T$ -equivariant cohomology algebra  $H_T^*(\Gamma_{\mathbf{i}}, \mathbb{C})$  has a  $\mathbb{C}$ -linear basis  $\{\sigma_K^T\}$ , where  $K$  runs over all subsets of  $[m]$ . Therefore, one defines the *equivariant Bott-Samelson coefficients*  $p_{K', K''}^{\mathbf{i}, K} \in S^{|K| - |K' \cup K''|}(V)$  similarly to (2.7) by:

$$(2.8) \quad \sigma_{K'}^T \cup \sigma_{K''}^T = \sum p_{K', K''}^{\mathbf{i}, K} \sigma_K^T,$$

where the summation is over all subsets  $K \subset [m]$  such that  $K' \cup K'' \subset K$  and  $|K| \leq |K'| + |K''|$ .

Note that  $p_{K', K''}^{\mathbf{i}, K} = \delta_{K, K' \cup K''}$  if  $K' \cap K'' = \emptyset$ .

The following result was proved by H. Duan in [12, Lemma 5.1] for the ordinary cohomology and by M. Willems in [56, Proposition 9] in the  $T$ -equivariant setting (see also our algebraic generalization, Theorem 3.18 and Corollary 4.20).

**Proposition 2.9.** *Let  $G$  be a Kac-Moody group and  $W$  be its Weyl group. Then for any sequence  $\mathbf{i} = (i_1, \dots, i_m) \in I^m$  one has:*

(a) *The pullback of the canonical projection  $\mu_{\mathbf{i}} : \Gamma_{\mathbf{i}} \rightarrow G/B$  is an algebra homomorphism  $\mu_{\mathbf{i}}^* : H_T^*(G/B, \mathbb{C}) \rightarrow H_T^*(\Gamma_{\mathbf{i}}, \mathbb{C})$  given by*

$$\mu_{\mathbf{i}}^*(\sigma_w^T) = \sum_{K \subset [m] : \mathbf{i}_K \in R(w)} \sigma_K^T$$

for all  $w \in W$  (with the convention that  $\mu_{\mathbf{i}}^*(\sigma_w) = 0$  if  $\mathbf{i}_K \notin R(w)$  for all  $k \subset [m]$ ).

(b) *If  $\mathbf{i} \in R(w)$  for some  $w \in W$ , then for any  $u, v \in W$  the equivariant Littlewood-Richardson and Bott-Samelson coefficients are related by:*

$$(2.9) \quad p_{u, v}^w = \sum p_{K', K''}^{\mathbf{i}, [m]},$$

with the summation over all subsets  $K', K'' \in [m]$  such that  $\mathbf{i}_{K'} \in R(u)$ ,  $\mathbf{i}_{K''} \in R(v)$ .

Thus, according to (2.9), in order to compute all  $p_{u, v}^w$ , it suffices to compute  $p_{K', K''}^{\mathbf{i}, K}$ . To do so, we need some notation.

For each bijection  $\varphi : L \xrightarrow{\sim} [m] \setminus M$  and any  $k \notin L$  we define a root  $\alpha_k^{(\varphi)} \in V$  by

$$(2.10) \quad \alpha_k^{(\varphi)} := \left( \prod_{r \in M <_k \cup \varphi(L <_k)} \overrightarrow{s_{i_r}} \right) (\alpha_{i_k}).$$

The following result, to the best of our knowledge, was previously unknown.

**Theorem 2.10.** *Let  $G$  be a Kac-Moody group and  $W = \langle s_i, i \in I \rangle$  be its Weyl group. Then (in the notation of Theorem 2.4) for each  $\mathbf{i} \in I^m$  and  $K, K', K'' \subset [m]$  with  $K' \cup K'' \subset K$ ,  $|K| \leq |K'| + |K''|$  one has:*

$$(2.11) \quad p_{K', K''}^{\mathbf{i}, K} = \sum p_{\varphi} \cdot \prod_{k \in (K' \cap K'') \setminus L} \alpha_k^{(\varphi)}$$

with the summation over all pairs  $(L, \varphi)$ , where

- $L$  is a subset of  $K' \cap K''$  such that  $|L| + |K' \cup K''| = |K|$ ;
- $\varphi : L \rightarrow K \setminus (K' \cup K'')$  is a bounded bijection.

We prove Theorem 2.10 in Section 5.

**Remark 2.11.** The Bott-Samelson coefficients  $c_{K', K''}^{\mathbf{i}, K} \in \mathbb{Z}$  for the ordinary cohomology  $H^*(\Gamma_{\mathbf{i}}(G), \mathbb{C})$  are given by

$$c_{K', K''}^{\mathbf{i}, K} = \delta_{|K|, |K'| + |K''|} p_{K', K''}^{\mathbf{i}, K}.$$

In this case, i.e., when  $|K| = |K'| + |K''|$ , the formula (2.11) simplifies because  $L = K' \cap K''$  and the thus summation in (2.11) is over all bounded bijections  $\varphi : K' \cap K'' \rightarrow K \setminus (K' \cup K'')$ .

**Remark 2.12.** The coefficients  $c_{K', K''}^{\mathbf{i}, K}$  were computed in [12] and  $p_{K', K''}^{\mathbf{i}, K}$  were computed for any  $K, K', K''$  in [57]. In the notation of Remark 2.8 one has:

$$(2.12) \quad c_{K', K''}^{\mathbf{i}, [m]} = \sum \prod_{1 \leq \ell \leq m} \frac{c_{*, \ell}!}{\prod_{1 \leq k < \ell} c_{k, \ell}!} \prod_{1 \leq k < \ell \leq m} b_{k, \ell}^{c_{k, \ell}},$$

where the summation is over all  $\mathbf{c}$  such that

$$c_{*, k} - c_{k, *} = \begin{cases} 1 & \text{if } k \in K' \cap K'' \\ -1 & \text{if } k \in K' \cup K'' \setminus (K' \cap K'') \\ 0 & \text{otherwise} \end{cases}$$

for  $k = 1, \dots, m$ .

Indeed, proving that the right hand sides of (2.11) and (2.12) are equal, is a rather interesting and challenging combinatorial problem.

Note that, unlike (equivariant) Littlewood-Richardson coefficients, the (equivariant) Bott-Samelson coefficients  $p_{K', K''}^{\mathbf{i}, K}$  are not always positive. Therefore, the formula (2.9) is not optimal. In fact, combining (2.11) with (2.9) (provided that  $\ell(u) + \ell(v) = \ell(w)$ ), we obtain the assertion of Theorem 2.4 with the  $\mathbf{i}$ -admissibility condition dropped. For instance, in the example of Remark 2.8, dropping  $\mathbf{i}$ -admissibility would (modestly) increase the number of summands in (2.3) from 19 to 190.

Instead of directly combining (2.11) with (2.9), we introduce (and compute) the “interpolating” coefficients as follows. For any triple of sequences  $\mathbf{i} \in I^m$ ,  $\mathbf{i}' \in I^{m'}$ ,  $\mathbf{i}'' \in I^{m''}$  define the *relative* ( $T$ -equivariant) Littlewood-Richardson coefficient  $p_{\mathbf{i}', \mathbf{i}''}^{\mathbf{i}}$  by

$$(2.13) \quad p_{\mathbf{i}', \mathbf{i}''}^{\mathbf{i}} = \sum p_{K', K''}^{\mathbf{i}, [m]}$$

where the summation is over all pairs  $K', K'' \subset [m]$  such that  $\mathbf{i}_{K'} = \mathbf{i}'$ ,  $\mathbf{i}_{K''} = \mathbf{i}''$  (in the notation of Definition 2.1).

We get that equation (2.9) simplifies to:

$$(2.14) \quad p_{u, v}^w = \sum p_{\mathbf{i}', \mathbf{i}''}^{\mathbf{i}}$$

for all  $u, v, w \in W$  and any given  $\mathbf{i} \in R(w)$ , where the summation is over all subsequences  $\mathbf{i}', \mathbf{i}''$  of  $\mathbf{i}$  such that  $\mathbf{i}' \in R(u)$ ,  $\mathbf{i}'' \in R(v)$ .

The following Theorem is our second main result which, taken together with (2.14), finishes the computation of all equivariant Littlewood-Richardson coefficients (and thus verifies Theorem 2.4).

**Theorem 2.13.** *Let  $G$  be a Kac-Moody group and  $W = \langle s_i, i \in I \rangle$  be its Weyl group. Then for each admissible sequence  $\mathbf{i} \in I^m$  one has:*

$$(2.15) \quad p_{\mathbf{i}', \mathbf{i}''}^{\mathbf{i}} = \sum_{\varphi} p_{\varphi} \cdot \prod_{k \in (K' \cap K'') \setminus L} \alpha_k^{(\varphi)}$$

with the summation over all quadruples  $(K', K'', L, \varphi)$ , where

- $K'$  and  $K''$  are subsets of  $[m]$  such that  $\mathbf{i}_{K'} = \mathbf{i}'$ ,  $\mathbf{i}_{K''} = \mathbf{i}''$ ;
- $L$  is a subset of  $K' \cap K''$  such that  $|L| + |K' \cup K''| = m$ ;
- $\varphi : L \rightarrow [m] \setminus (K' \cup K'')$  is an  $\mathbf{i}$ -admissible bounded bijection.

We prove Theorem 2.13 in Section 5. Note that the right hand side of (2.3) and (2.15) makes sense for any Coxeter group  $W$  acting on  $V$  by (2.1), even if the group  $G$  does not exist. The only data needed is the Cartan matrix  $A$ . In fact, the main ingredient of the proof is the realization of the relative coefficients  $p_{\mathbf{i}', \mathbf{i}''}^{\mathbf{i}}$  as the structure constants in the dual of the generalized nil-Hecke algebra of the free Coxeter semigroup as follows (see also Section 4).

**Theorem 2.14.** *For each Kac-Moody group  $G$  there exists a commutative  $S(V)$ -algebra  $\mathcal{A}(G)$  with the basis  $\{\sigma_{\mathbf{i}}\}$ , where  $\mathbf{i}$  runs over all sequences in  $I^m$ ,  $m \geq 0$  such that:*

- (a) *For any sequences  $\mathbf{i}' \in I^{m'}$  and  $\mathbf{i}'' \in I^{m''}$  one has:*

$$\sigma_{\mathbf{i}'} \sigma_{\mathbf{i}''} = \sum_{\mathbf{i}} p_{\mathbf{i}', \mathbf{i}''}^{\mathbf{i}} \sigma_{\mathbf{i}}$$

with the summation over all sequences  $\mathbf{i} \in I^m$ ,  $m \geq 0$  containing  $\mathbf{i}'$  and  $\mathbf{i}''$  as subsequences and such that  $m \leq m' + m''$ .

- (b) *The linear span of all  $\sigma_{\mathbf{i}}$  with admissible  $\mathbf{i}$  is a subalgebra  $\mathcal{A}^{\text{adm}}(G)$  of  $\mathcal{A}(G)$ .*

- (c) *The association  $\sigma_w^T \mapsto \sum_{\mathbf{i} \in R(w)} \sigma_{\mathbf{i}}$  defines an injective algebra homomorphism*

$$(2.16) \quad H_T^*(G/B) \hookrightarrow \mathcal{A}^{\text{adm}}(G).$$

- (d) *Given  $\mathbf{i} \in I^m$ , the association*

$$(2.17) \quad \sigma_{\mathbf{i}'} \mapsto \begin{cases} \sum_{K \subset [m]: \mathbf{i}_K = \mathbf{i}'} \sigma_K^T & \text{if } \mathbf{i}' \text{ is a subsequence of } \mathbf{i} \\ 0 & \text{otherwise} \end{cases}$$

for all  $\mathbf{i}' \in I^{m'}$ ,  $m' \geq 0$  defines an algebra homomorphism  $\mathcal{A}(G) \rightarrow H^*(\Gamma_{\mathbf{i}}(G), \mathbb{C})$ ; moreover, the canonical algebra homomorphism  $\varphi_{\mathbf{i}}^* : H_T^*(G/B, \mathbb{C}) \rightarrow H_T^*(\Gamma_{\mathbf{i}}, \mathbb{C})$  from Proposition 2.9(a) factors through as the composition of (2.16) and (2.17).



(e) For each  $\mathbf{i} \in I^m$  the linear span  $J_{\mathbf{i}} \subset \mathcal{A}(G)$  of all  $\sigma_{\mathbf{i}'}$  such that  $\mathbf{i}'$  is not a subsequence of  $\mathbf{i}$  is an ideal in  $\mathcal{A}(G)$ ; moreover, the kernel of (2.17) is  $J_{\mathbf{i}}$ , hence one has an injective homomorphism of algebras

$$(2.18) \quad \mathcal{A}(G)/J_{\mathbf{i}} \hookrightarrow H^*(\Gamma_{\mathbf{i}}(G), \mathbb{C})$$

We prove Theorem 2.14 in Section 4 (as a corollary of Proposition 4.16 and Theorem 4.19).

**Remark 2.15.** It follows from the results of Section 4 that the linear span of all  $\sigma_{\mathbf{i}'}$  with admissible  $\mathbf{i}'$  is an  $S(V)$ -subalgebra (which we denote  $\mathcal{A}^{adm}(G)$ ) of  $\mathcal{A}(G)$ .

Therefore, it would be natural to conjecture that for each  $\mathbf{i}$  there is a variety (or at least a topological space)  $X_{\mathbf{i}} = X_{\mathbf{i}}(G)$  and  $X_{\mathbf{i}}^{adm} = X_{\mathbf{i}}^{adm}(G)$  with the  $T$ -action and  $T$ -equivariant morphisms  $\Gamma_{\mathbf{i}}(G) \twoheadrightarrow X_{\mathbf{i}} \twoheadrightarrow X_{\mathbf{i}}^{adm} \rightarrow G/B$  such that:

- the canonical projection  $\mu_{\mathbf{i}} : \Gamma_{\mathbf{i}}(G) \rightarrow G/B$  factors through as  $X_{\mathbf{i}}$  and  $X_{\mathbf{i}}^{adm}$ .
- $H_T^*(X_{\mathbf{i}}^{adm}, \mathbb{C}) \cong \mathcal{A}_{\mathbf{i}}^{adm}(G)$ ,  $H_T^*(X_{\mathbf{i}}, \mathbb{C}) \cong \mathcal{A}_{\mathbf{i}}(G)$  and the algebra homomorphisms

$$H_T^*(G/B, \mathbb{C}) \rightarrow \mathcal{A}_{\mathbf{i}}^{adm}(G)/J_{\mathbf{i}}^{adm} \hookrightarrow \mathcal{A}_{\mathbf{i}}(G)/J_{\mathbf{i}} \hookrightarrow H^*(\Gamma_{\mathbf{i}}(G), \mathbb{C})$$

are just the pullbacks of the above morphisms  $\Gamma_{\mathbf{i}}(G) \twoheadrightarrow X_{\mathbf{i}} \twoheadrightarrow X_{\mathbf{i}}^{adm} \rightarrow G/B$  (here  $J_{\mathbf{i}}^{adm} = J_{\mathbf{i}} \cap \mathcal{A}^{adm}(G)$ ).

If  $\mathbf{i} \in R(w)$ , then  $X_{\mathbf{i}}^{adm}$  should be thought of as a “minimal resolution” of singularities of the corresponding Schubert variety in  $G/B$ .

We now apply Theorems 2.4 and 2.13 to give a combinatorial proof of positivity of the (equivariant) Littlewood-Richardson coefficients for a large class Kac-Moody groups. In [35], Kumar and Nori proved that if  $A$  is a Cartan matrix of some Kac-Moody group  $G$ , then every coefficient  $c_{u,v}^w \geq 0$ . This result for semisimple groups  $G$  is known via Kleiman’s transversality [21] and transitivity of  $G$ -action on the flag variety  $G/B$ . For equivariant coefficients corresponding to Kac-Moody groups, Graham in [17] proved that  $p_{u,v}^w$  have nonnegative coefficients as polynomials in the basis  $\{\alpha_i\}_{i \in I}$ . To the best of our knowledge, all known positivity proofs rely on the geometry of the flag variety  $G/B$ .

In Section 4 we introduce the notion of compatibility of a *quasi-Cartan* matrix  $A$ , i.e., an  $I \times I$ -matrix over  $\mathbb{k}$  such that  $a_{ii} = 2$  for  $i \in I$  and  $a_{ij} = 0 \Leftrightarrow a_{ji} = 0$ , and a Coxeter group  $W = \langle s_i, i \in I \rangle$  by requiring that  $W$  acts on the root space  $V = \bigoplus_{i \in I} \mathbb{k}\alpha_i$

by reflections defined in (2.1). We define the *generalized* Littlewood-Richardson coefficients  $p_{u,v}^w = p_{u,v}^w(A)$ ,  $c_{u,v}^w = \delta_{\ell(w), \ell(u) + \ell(v)} p_{u,v}^w$  for each such a compatible pair  $(A, W)$  and all relevant  $u, v, w \in W$ .

**Theorem 2.16.** *Let  $A$  be a quasi-Cartan matrix over  $\mathbb{R}$  compatible with a Coxeter group  $W$  such that*

$$(2.19) \quad a_{ij} < 0 \quad \text{and} \quad a_{ij}a_{ji} \geq 4$$

*for all  $i \neq j$ . Then all  $c_{u,v}^w$  are non-negative and all  $p_{u,v}^w \in \mathbb{R}_{\geq 0}[\alpha_i, i \in I]$ .*

The above theorem covers precisely those Kac-Moody groups  $G$  whose Weyl group  $W$  is a Coxeter group with no braid relations. We prove Theorem 2.16 in Section 6

by verifying that each factor of  $p_\varphi$  in (2.2) is nonnegative. This proof is completely combinatorial and relies on no geometry.

It is easy to show (see Section 7) that for each pair  $i \neq j$  the inequality  $c_{u,v}^w \geq 0$  for all  $u, v, w \in W_{ij} = \langle s_i, s_j \rangle$  is equivalent to:

$$(2.20) \quad a_{ij} \leq 0 \text{ and: either } a_{ij}a_{ji} \geq 4 \text{ or } a_{ij}a_{ji} = \left(2 \cos\left(\frac{\pi}{n_{ij}}\right)\right)^2$$

where  $n_{ij} \in \mathbb{Z}_{>0}$  is the order of  $s_i s_j$  in  $W_{ij} \subset W$ . In 1971 E. B. Vinberg proved in [55] that the condition (2.20) is equivalent to discreteness of the  $W$ -action on  $\mathbb{R}^I$ .

The following conjecture refines Theorem 2.16 and asserts that this necessary condition is also sufficient.

**Conjecture 2.17.** *Let  $A = (a_{ij})$  be a quasi-Cartan matrix such that (2.20) holds for all  $i \neq j$  (i.e.,  $W$  acts discretely on  $\mathbb{R}^I$ ). Then all Littlewood-Richardson coefficients  $c_{u,v}^w$  are nonnegative and all  $p_{u,v}^w \in \mathbb{R}_{\geq 0}[\alpha_i, i \in I]$ .*

Note that the quasi-Cartan matrices in the conjecture include all Cartan matrices of Kac-Moody groups and those involved in Theorem 2.16. In the case where  $W = \langle s_1, s_2 \rangle$  is a dihedral group of order  $2n$  and  $A$  is a  $2 \times 2$  symmetric matrix with  $a_{12} = a_{21} = 2 \cos(\frac{\pi}{n})$ , the nonnegativity of  $c_{u,v}^w$  has been verified by the first author and M. Kapovich in [3, Corollary 13.7].

We conclude Section 2 with the (yet conjectural) construction of (equivariant) *Littlewood-Richardson polynomials*  $p_{u,v}^w(\mathbf{A})$  and their strong positivity conjecture. Indeed, our definition of relative coefficients makes sense for the universal Coxeter group  $\widehat{W}$  generated by  $s_i, i \in I$  acting on  $\mathbb{Z}[\mathbf{A}]^I$ , where  $\mathbb{Z}[\mathbf{A}] = \mathbb{Z}[\mathbf{a}_{ij}, i \neq j]$  so that each  $p_{\mathbf{i}', \mathbf{i}''}^{\mathbf{i}}$  belong to  $\mathbb{Z}[\mathbf{A}, \alpha_i, i \in I]$ , i.e.,  $p_{\mathbf{i}', \mathbf{i}''}^{\mathbf{i}} = p_{\mathbf{i}', \mathbf{i}''}^{\mathbf{i}}(\mathbf{A})$  is a polynomial of the *universal* Cartan matrix  $\mathbf{A} = (\mathbf{a}_{ij})$  and all  $\alpha_i$  (i.e., it is a polynomial in  $|I|(|I| - 1) + |I| = |I|^2$  variables since  $\mathbf{a}_{ii} = 2$  for  $i \in I$ ).

Therefore, given any Coxeter group  $W$  generated by  $s_i, i \in I$  we define a polynomial  $p_{u,v}^{\mathbf{i}}(\mathbf{A})$  for any  $u, v \in W$  and any  $\mathbf{i} \in I^m$  (with  $\ell(u) + \ell(v) \geq m$ ) by the following analogue of (2.14):

$$(2.21) \quad p_{u,v}^{\mathbf{i}}(\mathbf{A}) = \sum p_{\mathbf{i}', \mathbf{i}''}^{\mathbf{i}}(\mathbf{A})$$

with the summation is over all sub-sequences  $\mathbf{i}', \mathbf{i}''$  of  $\mathbf{i}$  such that  $\mathbf{i}' \in R(u)$ ,  $\mathbf{i}'' \in R(v)$ . By the construction, if  $A$  is a (quasi-)Cartan matrix compatible with  $W$ , then  $p_{u,v}^{\mathbf{i}}(\mathbf{A})|_{\mathbf{A}=A} = p_{u,v}^w$  for all  $u, v, w \in W$  with  $\ell(u) + \ell(v) \geq \ell(w)$  and all  $\mathbf{i} \in R(w)$ .

We define the polynomial  $p_{u,v}^{\mathbf{i}}(t) \in \mathbb{k}[t, \alpha_i, i \in I]$  by the specialization

$$p_{u,v}^{\mathbf{i}}(t) := p_{u,v}^{\mathbf{i}}((1+t) \cdot A - 2t \cdot Id) ,$$

where  $Id$  is the  $I \times I$  identity matrix. By definition,  $p_{u,v}^{\mathbf{i}}(0) = p_{u,v}^w$  for each  $\mathbf{i} \in R(w)$ . Based on numerous examples (see Section 7), we expect a that stronger positivity result holds.

**Conjecture 2.18.** *For any  $A$  and  $W$  as in Conjecture 2.17 each polynomial  $p_{u,v}^{\mathbf{i}}(t)$  has nonnegative real coefficients.*

In fact, the coefficients of  $p_{u,v}^{\mathbf{i}}(t)$  belong to the sub-ring of  $\mathbb{R}$  generated by all  $a_{ij}$ , e.g., if  $A$  is an integer matrix, then the above conjecture asserts that all  $p_{u,v}^{\mathbf{i}}(t) \in \mathbb{Z}_{\geq 0}[t, \alpha_i, i \in I]$ . We verify the conjecture in Section 6 in the case when  $W$  is a free Coxeter group. The conjecture has also been verified by computer calculations for all  $p_{u,v}^{\mathbf{i}}(t)$  in finite types  $A_3$  and  $A_4$ .

The polynomials  $p_{u,v}^{\mathbf{i}}(t)$  depends on the choice  $\mathbf{i} \in R(w)$ , however, it frequently happens that  $p_{u,v}^{\mathbf{i}}(t) = p_{u,v}^{\mathbf{i}'}(t)$  for  $\mathbf{i}' \neq \mathbf{i}$ . Denote by  $\sim$  the equivalence relation on  $R(w)$  generated by pairs  $(\mathbf{i}, \mathbf{i}'')$  where  $\mathbf{i}'$  is obtained from  $\mathbf{i}$  by switching a single pair of adjacent indices  $i_k$  and  $i_{k+1}$  such that  $a_{i_k, i_{k+1}} = 0$ . We refer to this as the *commutativity* relation on  $R(w)$  and, following [50], we say that  $w \in W$  is *fully commutative* if  $R(w)$  is a single equivalence class.

**Conjecture 2.19.** *For any  $A$  and  $W$  as in Conjecture 2.17 we have for  $\mathbf{i}, \mathbf{i}' \in R(w)$  such that  $\mathbf{i} \sim \mathbf{i}'$ :*

$$p_{u,v}^{\mathbf{i}}(t) = p_{u,v}^{\mathbf{i}'}(t) .$$

*In particular, if  $w$  is a fully commutative element in  $W$ , then  $p_{u,v}^w(t)$  is well-defined.*

In particular, if  $\ell(w) = \ell(u) + \ell(v)$  and  $w$  is fully commutative, Conjectures 2.18 and 2.19 imply that  $c_{u,v}^w(t) := p_{u,v}^w(t)$  is a well-defined polynomial in  $t$  with nonnegative real coefficients.

If  $W$  is a free Coxeter group, then the conjecture trivially is true. We also verified the conjecture in finite types  $A_3$  and  $A_4$ .

### 3. TWISTED GROUP ALGEBRAS AND GENERALIZED LITTLEWOOD-RICHARDSON COEFFICIENTS

We begin with some facts on twisted group algebras. Let  $W$  be a monoid or a group and let  $Q$  be a commutative algebra over a field  $\mathbb{k}$ . For any covariant  $W$ -action on  $Q$  (i.e., such that  $w(q_1 \cdot q_2) = w(q_1) \cdot w(q_2)$ ) we define the *twisted* group (or, rather, monoidal) algebra  $Q_W := Q \rtimes \mathbb{k}W$  generated by  $Q$  and  $W$  subject to the relations:

$$wq = w(q) \cdot w$$

for all  $q \in Q$ ,  $w \in W$ .

We regard  $Q_W$  as a  $Q$ -module via the left multiplication. Note that one has a  $\mathbb{k}$ -linear isomorphism

$$\iota : \mathbb{k}W \otimes Q_W \xrightarrow{\sim} Q_W \bigotimes_Q Q_W$$

given by  $w \otimes qw' \mapsto w \otimes qw' = q(w \otimes w')$ . Taking into account that  $\mathbb{k}W \otimes Q_W$  is naturally a  $\mathbb{k}$ -algebra, this isomorphism turns  $Q_W \bigotimes_Q Q_W$  into an associative algebra as well. That is, the product in  $Q_W \bigotimes_Q Q_W$  is given by (cf. [31, Section 4.14]):

$$(q_1 w_1 \otimes q_2 w_2)(q'_1 w'_1 \otimes q'_2 w'_2) = (w_1 \otimes q_1 q_2 w_2)(w'_1 \otimes q'_1 q'_2 w'_2) = w_1 w'_1 \otimes q_1 q_2 w_2 q'_1 q'_2 w'_2$$

(note however, that in general the product in  $Q_W$  and in  $Q_W \bigotimes_Q Q_W$  is **not**  $Q$ -linear).

**Proposition 3.1.** *For any commutative module algebra  $Q$  over a monoid  $W$  one has:*

(a) *The algebra  $Q_W$  is a co-commutative coalgebra in the category of  $Q$ -modules with the coproduct  $\delta : Q_W \rightarrow Q_W \bigotimes_Q Q_W$  and the counit  $\varepsilon : Q_W \rightarrow Q$  given respectively by:*

$$\delta(qw) = q\delta(w) = w \otimes qw, \quad \varepsilon(qw) = q$$

for all  $q \in Q, w \in W$ .

(b) *The coproduct  $\delta$  from (a) is a homomorphism of algebras.*

(c) *For any  $x, y, z \in Q_W$  one has in the algebra  $Q_W \bigotimes_Q Q_W$ :*

$$(3.1) \quad \delta(x) \cdot (y \otimes z) = x_{(1)}y \otimes x_{(2)}z,$$

where  $\delta(x) = x_{(1)} \otimes x_{(2)}$  in the Sweedler notation.

*Proof.* Prove (a). First, verify that  $\delta$  is  $Q$ -linear. Indeed,

$$\delta(q_1q_2w) = w \otimes q_1q_2w = q_1w \otimes q_2w = q_1(w \otimes q_2w) = q_1\delta(q_2w).$$

Furthermore, the identity

$$(\delta \otimes 1)\delta(qw) = \delta(w) \otimes qw = w \otimes w \otimes qw = w \otimes \delta(qw) = (1 \otimes \delta)\delta(qw)$$

verifies the coassociativity of  $\delta$ . Now verify the counit axiom:

$$(\varepsilon \otimes 1)\delta(qw) = (\varepsilon \otimes 1)(w \otimes qw) = qw = (1 \otimes \varepsilon)(qw) = (1 \otimes \varepsilon)\delta(qw).$$

Finally, let us verify the co-commutativity. Let  $\tau : Q_W \bigotimes_Q Q_W$  be the permutation of factors. Then

$$\tau\delta(qw) = \tau(w \otimes qw) = qw \otimes w = w \otimes qw = \delta(qw).$$

This proves (a).

Prove (b) now. Indeed,

$$\begin{aligned} \delta((q_1w_1)(q_2w_2)) &= \delta((q_1w_1(q_2))w_1w_2) = w_1w_2 \otimes (q_1w_1(q_2))w_1w_2 \\ &= w_1w_2 \otimes q_1w_1q_2w_2 = (w_1 \otimes q_1w_1)(w_2 \otimes q_2w_2) = \delta(q_1w_1)\delta(q_2w_2). \end{aligned}$$

This proves (b).

Prove (c) now. Indeed, it suffices to verify (3.1) for  $x = q_1w_1, y = q_2w_2, z = q_3w_3$ :

$$\begin{aligned} \delta(q_1w_1) \cdot (q_2w_2 \otimes q_3w_3) &= (w_1 \otimes q_1w_1)(w_2 \otimes q_2q_3w_3) = w_1w_2 \otimes q_1w_1q_2q_3w_3 \\ &= w_1(q_2)w_1w_2 \otimes q_1w_1q_3w_3 = w_1q_2w_2 \otimes q_1w_1q_3w_3 = w_1y \otimes q_1w_1z. \end{aligned}$$

Part (c) is proved.

Therefore, the proposition is proved.  $\square$

**Remark 3.2.** Note that  $Q_W$  is not a bialgebra in the category of  $Q$ -modules because neither  $Q_W$  nor  $Q_W \bigotimes_Q Q_W$  is not an algebra in this category.

Clearly, if  $M$  and  $N$  are free  $Q$ -modules, and  $B_M, B_N$  are bases respectively in  $M$  and  $N$ , then the set  $B_M \otimes B_N = \{b \otimes b' \mid b \in B_M, b' \in B_N\}$  is a basis of  $M \underset{Q}{\otimes} N$ .

In particular, if  $B$  is a basis of  $Q_W$ , then set  $B \otimes B \cong B \times B$  is a basis of  $Q_W \underset{Q}{\otimes} Q_W$ .

Using this, for each basis  $B = \{x_w, w \in W\}$  of  $Q_W$ , we define *generalized Littlewood-Richardson coefficients*  $p_{u,v}^w \in Q$  by the formula:

$$(3.2) \quad \delta(x_w) = \sum_{u,v \in W} p_{uv}^w x_u \otimes x_v .$$

Dualizing this definition, we obtain the following result.

**Proposition 3.3.** *Let  $f : Q \rightarrow Q'$  be a homomorphism of commutative  $\mathbb{k}$ -algebras such that the set  $\{w \in W : f(p_{u,v}^w) \neq 0\}$  is finite for all  $u, v \in W$ . Then there is a unique (associative) commutative  $Q'$ -algebra  $\mathcal{A}_f$  with the free  $Q'$ -basis  $\{\sigma_w \mid w \in W\}$  and the following multiplication table:*

$$\sigma_u \sigma_v = \sum_{w \in W} f(p_{u,v}^w) \sigma_w$$

for all  $u, v \in W$ .

*Proof.* We need the following result.

**Lemma 3.4.** *Let  $\delta : \mathcal{C} \rightarrow \mathcal{C} \underset{Q}{\otimes} \mathcal{C}$  be a coalgebra in the category of  $Q$ -modules. Assume that  $B$  is a basis of  $\mathcal{C}$  such that*

$$\delta(b) = \sum_{b', b'' \in B} p_{b', b''}^b b \otimes b' ,$$

where all  $p_{b', b''}^b \in Q$ . Then for any homomorphism  $f : Q \rightarrow Q'$  of commutative  $\mathbb{k}$ -algebras such that the set  $\{b \in B : f(p_{b', b''}^b) \neq 0\}$  is finite for all  $b', b'' \in B$  there is a unique associative  $Q'$ -algebra  $\mathcal{A} = \mathcal{A}_f$  with the basis  $\{\sigma_b \mid b \in B\}$  and the following multiplication table:

$$\sigma_{b'} \sigma_{b''} = \sum_{b \in B} f(p_{b', b''}^b) \sigma_b$$

for all  $b', b'' \in B$ . If, additionally,  $\mathcal{C}$  was co-commutative, then  $\mathcal{A}_f$  is commutative.

*Proof.* Indeed,

$$\begin{aligned} (\delta \otimes 1)\delta(b_1) &= \sum_{b, b_4 \in B} p_{b, b_4}^{b_1} \delta(b) \otimes b_4 = \sum_{b, b_4} p_{b, b_4}^{b_1} \left( \sum_{b_2, b_3 \in B} p_{b_2, b_3}^b b_2 \otimes b_3 \right) \otimes b_4 \\ &= \sum_{b, b_2, b_3, b_4 \in B} p_{b, b_4}^{b_1} p_{b_2, b_3}^b b_2 \otimes b_3 \otimes b_4 . \\ (1 \otimes \delta)\delta(b_1) &= \sum_{b, b_2 \in B} p_{b_2, b}^{b_1} b_2 \otimes \delta(b) = \sum_{b, b_2} p_{b_2, b}^{b_1} b_2 \otimes \left( \sum_{b_3, b_4 \in B} p_{b_3, b_4}^b b_3 \otimes b_4 \right) \\ &= \sum_{b, b_2, b_3, b_4 \in B} p_{b_2, b}^{b_1} p_{b_3, b_4}^b b_2 \otimes b_3 \otimes b_4 . \end{aligned}$$

Taking into the account that  $B \otimes B \otimes B \cong B \times B \times B$  is the basis of  $\mathcal{C} \underset{Q}{\otimes} \mathcal{C} \underset{Q}{\otimes} \mathcal{C}$ , the coassociativity of  $\delta$  implies

$$\sum_{b \in B} p_{b,b_4}^{b_1} p_{b_2,b_3}^b = \sum_{b \in B} p_{b_2,b}^{b_1} p_{b_3,b_4}^b$$

for all  $b_1, b_2, b_3, b_4 \in B$ . Applying  $f$ , this implies that

$$\begin{aligned} (\sigma_{b_2} \sigma_{b_3}) \sigma_{b_4} &= \sum_{b \in B} f(p_{b_2,b_3}^b) \sigma_b \sigma_{b_4} = \sum_{b, b_1 \in B} f(p_{b_2,b_3}^b p_{b,b_4}^{b_1}) \sigma_{b_1} = \sum_{b, b_1 \in B} f(p_{b_2,b}^{b_1} p_{b_3,b_4}^b) \sigma_{b_1} \\ &= \sum_{b \in B} f(p_{b_3,b_4}^b) \sigma_{b_2} \sigma_b = \sigma_{b_2} (\sigma_{b_3} \sigma_{b_4}) . \end{aligned}$$

Finally, note that co-commutativity of  $\mathcal{C}$  is equivalent to  $\tau\delta(b) = \delta(b)$  for all  $b \in B$ , i.e.,

$$\sum_{b', b''} b'' \otimes p_{b', b''}^b b' = \sum_{b', b''} p_{b', b''}^b b' \otimes b'' ,$$

i.e.,  $p_{b'', b'}^b = p_{b', b''}^b$  for all  $b, b', b'' \in B$ . This implies that  $\mathcal{A}_f$  is commutative. The lemma is proved.  $\square$

Taking  $\mathcal{C} = Q_W$ ,  $B = \{x_w, w \in W\}$ , we finish the proof of Proposition 3.3.  $\square$

In the assumptions of Proposition 3.3 let  $\langle \cdot, \cdot \rangle : \mathcal{A}_f \times Q_W \rightarrow Q'$  be the  $Q'$ -linear pairing given by

$$(3.3) \quad \langle q' \sigma_u, q x_v \rangle = \delta_{u,v} \cdot q' f(q)$$

for all  $u, v \in W$ ,  $q \in Q$ ,  $q' \in Q'$ .

**Corollary 3.5.** *In the assumptions of Proposition 3.3, we have*

(a) *The pairing (3.3) satisfies:*

$$\langle ab, x \rangle = \langle a \otimes b, \delta(x) \rangle = \langle a, x_{(1)} \rangle \langle b, x_{(2)} \rangle$$

for all  $a, b \in \mathcal{A}_f$ ,  $x \in Q_W$ , where  $\delta(x) = x_{(1)} \otimes x_{(2)}$  in the Sweedler notation.

(b) *For each  $w \in W$  the assignment  $a \mapsto \langle a, w \rangle$ ,  $a \in \mathcal{A}_f$  is a  $Q'$ -algebra homomorphism*

$$\xi_w : \mathcal{A}_f \rightarrow Q'$$

*Proof.* Prove (a). It suffices to verify the identity for  $a = \sigma_u$ ,  $b = \sigma_v$ ,  $x = x_w$ . Indeed,

$$\begin{aligned} \langle \sigma_u \otimes \sigma_v, \delta(x_w) \rangle &= \langle \sigma_u \otimes \sigma_v, \sum_{u', v' \in W} p_{u', v'}^w x_{u'} \otimes x_{v'} \rangle = \sum_{u', v'} f(p_{u', v'}^w) \langle \sigma_u, x_{u'} \rangle \langle \sigma_v, x_{v'} \rangle \\ &= \sum_{u', v'} f(p_{u', v'}^w) \delta_{u, u'} \delta_{v, v'} = \langle \sum_{w'} p_{u, v}^w \delta_{w, w'} \sigma_{w'}, x_w \rangle = \langle \sigma_u \sigma_v, x_w \rangle . \end{aligned}$$

This proves (a).

Prove (b). The  $Q'$ -linearity of  $\xi_w$  is obvious. Prove that  $\xi_w$  respects multiplication. Indeed, for all  $w \in W$ ,  $a, b \in \mathcal{A}_f$  we have

$$\begin{aligned} \xi_w(ab) &= \langle ab, w \rangle = \langle a \otimes b, \delta(w) \rangle = \langle a \otimes b, w \otimes w \rangle \\ &= \langle a, w \rangle \langle b, w \rangle = \xi_w(a) \xi_w(b) . \end{aligned}$$

This proves (b).

The corollary is proved.  $\square$

In what follows (Proposition 3.8), we introduce the analogues of  $p_{uv}^w$  which we refer to as *relative* (generalized) Littlewood-Richardson coefficients.

**Definition 3.6.** Given a subset  $S = \{s_i, i \in I\}$  of  $W \setminus \{1\}$ , we say that a subset  $X = \{x_i, i \in I\}$  of  $Q_W$  is *S-tame* if  $X$  is a basis of the (free)  $Q$ -module  $\sum_{i \in I} Q(s_i - 1)$ .

For an *S-tame* set  $X$  we have:

$$(3.4) \quad x_i = \sum_{j \in I} r_{ij}(s_j - 1) \quad \text{and} \quad s_i = 1 + \sum_{j \in I} q_{ij}x_j$$

for some mutually inverse  $I \times I$  matrices  $(q_{ij})$  and  $(r_{ij})$  over  $Q$ .

For any sequence  $\mathbf{i} := (i_1, \dots, i_m) \in I^m$ , define a monomial  $x_{\mathbf{i}} \in Q_W$  by:

$$x_{\mathbf{i}} := x_{i_1} \cdots x_{i_m}$$

with the convention that  $x_{\emptyset} = 1$ .

The following fact is obvious.

**Lemma 3.7.** *There is a unique left action of  $Q_W$  on  $Q$  such that*

$$(qw)(q') = q \cdot w(q')$$

for  $q, q' \in Q$ ,  $w \in W$ . The  $Q_W$ -action on  $Q$  satisfies for all  $x \in Q_W$ :

$$x(q) = \varepsilon(xq) ,$$

where  $\varepsilon : Q_W \rightarrow Q$  is the counit from Proposition 3.1(a).

The following result is a generalization of Kostant-Kumar recursion from [31]

**Proposition 3.8.** *For any S-tame set  $X = \{x_i, i \in I\}$  in  $Q_W$  we have:*

$$(3.5) \quad \delta(x_{\mathbf{i}}) = \sum_{\mathbf{i}', \mathbf{i}''} p_{\mathbf{i}', \mathbf{i}''}^{\mathbf{i}} x_{\mathbf{i}'} \otimes x_{\mathbf{i}''}$$

where the summation is over all pairs of sequences  $(\mathbf{i}', \mathbf{i}'') \in I^{m'} \times I^{m''}$  with  $m', m'' \leq m$  and the coefficients  $p_{\mathbf{i}', \mathbf{i}''}^{\mathbf{i}}$  are determined recursively by  $p_{\emptyset, \emptyset}^{\emptyset} = 1$  and:

$$(3.6) \quad p_{\mathbf{i}', \mathbf{i}''}^{\mathbf{i}} = x_{i_1}(p_{\mathbf{i}', \mathbf{i}''}^{\tilde{\mathbf{i}}}) + \sum_{j \in I} r_{i_1, j} \left( q_{j, i_1'} s_j(p_{\mathbf{i}', \mathbf{i}''}^{\tilde{\mathbf{i}}}) + q_{j, i_1''} s_j(p_{\mathbf{i}', \mathbf{i}''}^{\tilde{\mathbf{i}}}) + q_{j, i_1'} q_{j, i_1''} s_j(p_{\mathbf{i}', \mathbf{i}''}^{\tilde{\mathbf{i}}}) \right) ,$$

if  $m \geq 1$ , where  $\tilde{\mathbf{i}}$  stands for a sequence obtained from  $\mathbf{i}$  by deleting the first entry  $i_1$ .

*Proof.* First, compute  $\delta(x_i)$ . Indeed, using (3.4), we obtain:

$$\begin{aligned} \delta(x_i) &= \sum_{j \in I} r_{ij}(s_j \otimes s_j - 1 \otimes 1) = \sum_{j \in I} r_{ij}((s_j - 1) \otimes 1 + s_j \otimes (s_j - 1)) \\ &= x_i \otimes 1 + \sum_{j, i' \in I} r_{ij} q_{j, i'} s_j \otimes x_{i'} = x_i \otimes 1 + 1 \otimes x_i + \sum_{j, i', i'' \in I} r_{ij} q_{j, i'} q_{j, i''} x_{i'} \otimes x_{i''} . \end{aligned}$$

We need the following result.

**Lemma 3.9.** *For each  $i \in I$ ,  $p \in Q$  we have:*

$$x_i p = x_i(p) + \sum_{j, i'} r_{ij} q_{j, i'} s_j(p) x_{i'} .$$

*Proof.* Indeed,

$$x_i p = \sum_j r_{ij} (s_j - 1) p = \sum_j r_{ij} ((s_j - 1)(p) + s_j(p)(s_j - 1)) = x_i(p) + \sum_{j, i'} r_{ij} s_j(p) q_{j, i'} x_{i'} .$$

□

Furthermore, for  $\mathbf{i} = (i_1, \dots, i_m) \in I^m$  denote  $\tilde{\mathbf{i}} = \mathbf{i} \setminus \{i_1\} := (i_2, \dots, i_m)$  so that  $x_{\mathbf{i}} = x_i x_{\tilde{\mathbf{i}}}$ . Therefore, using the inductive hypothesis in the form:

$$\delta(x_{\mathbf{i}}) = \sum_{\tilde{\mathbf{i}}', \tilde{\mathbf{i}}''} p_{\tilde{\mathbf{i}}', \tilde{\mathbf{i}}''}^{\tilde{\mathbf{i}}} x_{\tilde{\mathbf{i}}'} \otimes x_{\tilde{\mathbf{i}}''} ,$$

we obtain using the above computation of  $\delta(x_i)$ , Proposition 3.1(c), and Lemma 3.9:

$$\begin{aligned} \delta(x_{\mathbf{i}}) &= \delta(x_{i_1}) \delta(x_{\tilde{\mathbf{i}}}) = (x_{i_1} \otimes 1 + \sum_{j, i_1'} r_{i_1, j} q_{j, i_1'} s_j \otimes x_{i_1'}) \sum_{\tilde{\mathbf{i}}', \tilde{\mathbf{i}}''} p_{\tilde{\mathbf{i}}', \tilde{\mathbf{i}}''}^{\tilde{\mathbf{i}}} x_{\tilde{\mathbf{i}}'} \otimes x_{\tilde{\mathbf{i}}''} \\ &= \sum_{\tilde{\mathbf{i}}', \tilde{\mathbf{i}}''} x_{i_1} p_{\tilde{\mathbf{i}}', \tilde{\mathbf{i}}''}^{\tilde{\mathbf{i}}} x_{\tilde{\mathbf{i}}'} \otimes x_{\tilde{\mathbf{i}}''} + \sum_{j, i_1', \tilde{\mathbf{i}}', \tilde{\mathbf{i}}''} r_{i_1, j} q_{j, i_1'} s_j p_{\tilde{\mathbf{i}}', \tilde{\mathbf{i}}''}^{\tilde{\mathbf{i}}} x_{\tilde{\mathbf{i}}'} \otimes x_{i_1'} x_{\tilde{\mathbf{i}}''} \\ &= \sum_{\tilde{\mathbf{i}}', \tilde{\mathbf{i}}''} x_{i_1} p_{\tilde{\mathbf{i}}', \tilde{\mathbf{i}}''}^{\tilde{\mathbf{i}}} x_{\tilde{\mathbf{i}}'} \otimes x_{\tilde{\mathbf{i}}''} + \sum_{j, i_1', \tilde{\mathbf{i}}', \tilde{\mathbf{i}}''} r_{i_1, j} q_{j, i_1'} (s_j(p_{\tilde{\mathbf{i}}', \tilde{\mathbf{i}}''}^{\tilde{\mathbf{i}}}) + p_{\tilde{\mathbf{i}}', \tilde{\mathbf{i}}''}^{\tilde{\mathbf{i}}} (s_j - 1)) x_{\tilde{\mathbf{i}}'} \otimes x_{i_1'} x_{\tilde{\mathbf{i}}''} \\ &= \sum_{\tilde{\mathbf{i}}', \tilde{\mathbf{i}}''} \left( x_{i_1} (p_{\tilde{\mathbf{i}}', \tilde{\mathbf{i}}''}^{\tilde{\mathbf{i}}}) x_{\tilde{\mathbf{i}}'} \otimes x_{\tilde{\mathbf{i}}''} + \sum_{j, i_1'} r_{i_1, j} q_{j, i_1'} s_j (p_{\tilde{\mathbf{i}}', \tilde{\mathbf{i}}''}^{\tilde{\mathbf{i}}}) x_{i_1'} x_{\tilde{\mathbf{i}}'} \otimes x_{\tilde{\mathbf{i}}''} \right. \\ &\quad \left. + \sum_{j, i_1'} r_{i_1, j} q_{j, i_1'} s_j (p_{\tilde{\mathbf{i}}', \tilde{\mathbf{i}}''}^{\tilde{\mathbf{i}}}) x_{\tilde{\mathbf{i}}'} \otimes x_{i_1'} x_{\tilde{\mathbf{i}}''} + \sum_{j, i_1', i_1''} r_{i_1, j} q_{j, i_1'} s_j (p_{\tilde{\mathbf{i}}', \tilde{\mathbf{i}}''}^{\tilde{\mathbf{i}}}) q_{j, i_1'} x_{i_1'} x_{\tilde{\mathbf{i}}'} \otimes x_{i_1''} x_{\tilde{\mathbf{i}}''} \right) \\ &= \sum_{\tilde{\mathbf{i}}', \tilde{\mathbf{i}}''} p_{\tilde{\mathbf{i}}', \tilde{\mathbf{i}}''}^{\mathbf{i}} x_{\tilde{\mathbf{i}}'} \otimes x_{\tilde{\mathbf{i}}''} . \end{aligned}$$

This proves (3.5). Therefore, Proposition 3.8 is proved. □

We refer to  $p_{\tilde{\mathbf{i}}', \tilde{\mathbf{i}}''}^{\mathbf{i}}$  as the *relative Littlewood-Richardson coefficients*. Since the  $x_{\mathbf{i}}$  are not linearly independent in general, the relative Littlewood-Richardson are not unique. Nevertheless, we can restore the uniqueness by replacing  $W$  with a larger monoid as follows.

**Theorem 3.10.** (*Folding principle*) *Let  $Q$  (resp.  $\widehat{Q}$ ) be a commutative module algebra over a monoid  $W$  (resp.  $\widehat{W}$ ). Let  $\varphi_- : \widehat{W} \rightarrow W$  be a homomorphism of monoids and let  $\varphi_+ : \widehat{Q} \rightarrow Q$  be an algebra homomorphism commuting with the  $\widehat{W}$ -action. Then:*

(a) *there exists a unique algebra homomorphism  $\varphi : \widehat{Q}_{\widehat{W}} \rightarrow Q_W$  such that*

$$\varphi|_{1 \times \widehat{W}} = \varphi_-, \quad \varphi|_{\widehat{Q} \times 1} = \varphi_+$$



and the following diagram is commutative:

$$(3.7) \quad \begin{array}{ccc} \widehat{Q}_{\widehat{W}} & \xrightarrow{\widehat{\delta}} & \widehat{Q}_{\widehat{W}} \otimes_{\widehat{Q}} \widehat{Q}_{\widehat{W}} \\ \varphi \downarrow & & \downarrow \varphi \otimes \varphi \\ Q_W & \xrightarrow{\delta} & Q_W \otimes_Q Q_W \end{array}$$

(b) For any  $S$ -tame set  $X = \{x_i, i \in I\}$  in  $Q_W$ , any  $\widehat{S}$ -tame set  $\widehat{X} = \{\widehat{x}_k, k \in K\}$  in  $\widehat{Q}_{\widehat{W}}$ , and a map  $\pi : K \rightarrow I$  such that

$$(3.8) \quad \varphi(\widehat{x}_k) = x_{\pi(k)}$$

for all  $k \in K$  one has (for all  $\mathbf{i} \in I^m$ ,  $\mathbf{i}' \in I^{m'}$ ,  $\mathbf{i}'' \in I^{m''}$  with  $m', m'' \leq m$ ):

$$(3.9) \quad p_{\mathbf{i}', \mathbf{i}''}^{\mathbf{i}} = \sum \varphi(\widehat{p}_{\mathbf{j}', \mathbf{j}''}^{\mathbf{j}}),$$

where  $\mathbf{j} \in K^m$  is any sequence such that  $\pi(\mathbf{j}) = \mathbf{i}$  and the summation is over all sequences  $\mathbf{j}' \in K^{m'}$ ,  $\mathbf{j}'' \in K^{m''}$  such that  $\pi(\mathbf{j}') = \mathbf{i}'$ ,  $\pi(\mathbf{j}'') = \mathbf{i}''$ , where  $\widehat{p}_{\mathbf{j}', \mathbf{j}''}^{\mathbf{j}}$  are relative Littlewood-Richardson coefficients for  $\widehat{Q}_{\widehat{W}}$ .

*Proof.* Prove (a). We verify the first assertion. Define a linear map  $\varphi : \widehat{Q}_{\widehat{W}} \rightarrow Q_W$  by:

$$\varphi(\widehat{q}\widehat{w}) = \varphi_+(\widehat{q})\varphi_-(\widehat{w}).$$

In order to prove that  $\varphi$  is an algebra homomorphism it suffices to show that  $\varphi(\widehat{w}\widehat{q}) = \varphi(\widehat{w})\varphi(\widehat{q})$  for all  $\widehat{q} \in \widehat{Q}$ ,  $\widehat{w} \in \widehat{W}$ . Indeed,

$$\begin{aligned} \varphi(\widehat{w}\widehat{q}) &= \varphi(\widehat{w}(\widehat{q}) \cdot w) = \varphi_+(\widehat{w}(\widehat{q})) \cdot \varphi_-(\widehat{w}) \\ &= (\varphi_-(\widehat{w}))(\varphi_+(\widehat{q}))\varphi_-(\widehat{w}) = \varphi_-(\widehat{w}) \cdot \varphi_+(\widehat{q}) = \varphi(\widehat{w})\varphi(\widehat{q}). \end{aligned}$$

Now verify the commutativity of the diagram (3.7). Indeed,

$$\delta(\varphi(\widehat{q}\widehat{w})) = \delta(\varphi(\widehat{q})\varphi(\widehat{w})) = \varphi(\widehat{w}) \otimes \varphi(\widehat{q}\widehat{w}) = (\varphi \otimes \varphi)(\widehat{w} \otimes \widehat{q}\widehat{w}) = (\varphi \otimes \varphi)\widehat{\delta}(\widehat{q}\widehat{w}).$$

This proves (a).

Prove (b) now. We need the following result.

**Lemma 3.11.** *Let  $\widehat{W}$  be the free monoid generated by  $S \subset W$ , then:*

(i) *One has a (unique) algebra homomorphism  $\varphi : Q_{\widehat{W}} \rightarrow Q_W$  such that  $\varphi|_S = Id_S$  and  $\varphi|_Q = Id_Q$ ;*

(ii) *for any  $S$ -tame set  $X = \{x_i \in I\}$  in  $Q_W$  the set*

$$\widehat{X} = \{\widehat{x}_i = \varphi^{-1}(x_i) \cap \sum_{s \in S} Q \cdot (s - 1), i \in I\}$$

*is  $S$ -tame in  $Q_{\widehat{W}}$ ;*

(iii) *The monomials  $\widehat{x}_{\mathbf{i}} = \widehat{x}_{i_1} \cdots \widehat{x}_{i_m}$  are  $Q$ -linearly independent in  $Q_{\widehat{W}}$ .*

(iv) Each relative Littlewood-Richardson coefficient  $\widehat{p}_{\mathbf{i}', \mathbf{i}''}^{\mathbf{i}}$  for  $Q_{\widehat{W}}$  with respect to  $\widehat{X}$  equals to the relative Littlewood-Richardson coefficient  $p_{\mathbf{i}', \mathbf{i}''}^{\mathbf{i}}$  for  $Q_W$  and is uniquely determined by the expansion (3.5):

$$(3.10) \quad \widehat{\delta}(\widehat{x}_{\mathbf{i}}) = \sum_{\mathbf{i}', \mathbf{i}''} p_{\mathbf{i}', \mathbf{i}''}^{\mathbf{i}} \widehat{x}_{\mathbf{i}'} \otimes \widehat{x}_{\mathbf{i}''}.$$

*Proof.* Indeed,  $\varphi : Q_{\widehat{W}} \rightarrow Q_W$  as an algebra homomorphism by Theorem 3.10(a). This verifies (i). Furthermore, since the restriction of  $\varphi$  to  $\sum_{s \in S} Q \cdot (s - 1)$  is the identity map, one can trivially lift each  $x_i \in X$  to a unique element  $\widehat{x}_i \in Q_{\widehat{W}}$  such that  $\varphi(\widehat{x}_i) = x_i$ . This verifies (ii). Let us show that all monomials  $x_{\mathbf{i}}$  form a basis in the subalgebra  $Q_{\widehat{W}_+}$  of  $Q_{\widehat{W}}$  generated by  $S$  and  $Q$ . Indeed,  $Q_{\widehat{W}_+}$  has a  $Q$ -basis of the form  $w_{\mathbf{i}} = s_{i_1} \cdots s_{i_m}$ , where  $\mathbf{i} \in I^m$ ,  $m \geq 0$  runs over all sequences. Denote by  $\mathcal{A}_{\leq n}$  the  $Q$ -submodule of  $Q_{\widehat{W}_+}$  spanned by all  $w_{\mathbf{i}}$ ,  $\mathbf{i} \in I^m$ ,  $m \leq n$ . Also denote by  $\mathcal{B}_{\leq n}$  the  $Q$ -submodule of  $Q_{\widehat{W}_+}$  spanned by all  $x_{\mathbf{i}}$ ,  $\mathbf{i} \in I^m$ ,  $m \leq n$ . Let us show that  $\mathcal{A}_{\leq n} = \mathcal{B}_{\leq n}$ . Clearly, both  $\mathcal{A}_{\leq n}$  defines a filtration on the algebra  $Q_{\widehat{W}_+}$  such that  $\mathcal{A}_{\leq n} = (\mathcal{A}_{\leq 1})^n$ . Note that

$$x_i q \in Q + \sum_{j \in I} Q \cdot (s_j - 1) \subseteq Q + \sum_{j \in I} Q \cdot x_j = \mathcal{B}_{\leq 1}$$

for each  $i \in I$ ,  $q \in Q$ . This implies that  $\mathcal{B}_{\leq n}$  is also a filtration on the algebra  $Q_{\widehat{W}_+}$  such that  $\mathcal{B}_{\leq n} = (\mathcal{B}_{\leq 1})^n$ . Since  $\mathcal{A}_{\leq 1} = \mathcal{B}_{\leq 1}$  by definition of the tame set  $\widehat{X}$ , we see that  $\mathcal{A}_{\leq n} = \mathcal{B}_{\leq n}$ . This proves linear independence of all  $x_{\mathbf{i}}$  and, thus, verifies part (iii). Finally, in view of (iii), the coefficients  $p_{\mathbf{i}', \mathbf{i}''}^{\mathbf{i}}$  are uniquely determined by:

$$\widehat{\delta}(\widehat{x}_{\mathbf{i}}) = \sum_{\mathbf{i}', \mathbf{i}''} \widehat{p}_{\mathbf{i}', \mathbf{i}''}^{\mathbf{i}} \widehat{x}_{\mathbf{i}'} \otimes \widehat{x}_{\mathbf{i}''}.$$

This implies that  $\widehat{p}_{\mathbf{i}', \mathbf{i}''}^{\mathbf{i}} = p_{\mathbf{i}', \mathbf{i}''}^{\mathbf{i}}$  for all relevant  $\mathbf{i}, \mathbf{i}', \mathbf{i}''$  because both families  $\{p_{\mathbf{i}', \mathbf{i}''}^{\mathbf{i}}\}$  and  $\{\widehat{p}_{\mathbf{i}', \mathbf{i}''}^{\mathbf{i}}\}$  satisfy the same recursion (3.4). This verifies (iv).

The lemma is proved.  $\square$

Furthermore, we prove (3.9). Using Lemma 3.11, without loss of generality we may assume that  $W$  is a free monoid generated by  $S = \{s_i, i \in I\}$  and  $\widehat{W}$  is a free monoid generated by  $\widehat{S} = \{\widehat{s}_1, \dots, \widehat{s}_m\}$ . In particular, one has a unique expansion

$$\delta(x_{\mathbf{i}}) = \sum_{\mathbf{i}', \mathbf{i}''} p_{\mathbf{i}', \mathbf{i}''}^{\mathbf{i}} x_{\mathbf{i}'} \otimes x_{\mathbf{i}''}$$

where the summation is over all sequences  $\mathbf{i}' \in I^{m'}$  and  $\mathbf{i}'' \in I^{m''}$ ,  $m', m'' \leq m$  and

$$\widehat{\delta}(\widehat{x}_{\mathbf{j}}) = \sum_{\mathbf{j}', \mathbf{j}''} \widehat{p}_{\mathbf{j}', \mathbf{j}''}^{\mathbf{j}} \widehat{x}_{\mathbf{j}'} \otimes \widehat{x}_{\mathbf{j}''},$$

where the summation is over all sequences  $\mathbf{j}' \in K^{m'}$  and  $\mathbf{j}'' \in K^{m''}$ ,  $m', m'' \leq m$ . Since the diagram (3.7) is commutative, we obtain:

$$\delta(\varphi(\widehat{x}_{\mathbf{j}})) = (\varphi \otimes \varphi)(\widehat{\delta}(\widehat{x}_{\mathbf{j}}))$$

Since  $\widehat{\varphi}(\widehat{x}_{\mathbf{j}'}) = x_{\pi(\mathbf{j}')} for any  $\mathbf{j}' \in K^{m'}$ , we obtain:$

$$\delta(x_{\mathbf{i}}) = \sum_{\mathbf{j}', \mathbf{j}''} \varphi(\widehat{p}_{\mathbf{j}', \mathbf{j}''}^{\mathbf{j}}) x_{\varphi(\mathbf{j}')} \otimes x_{\varphi(\mathbf{j}'')} .$$

Since the tensors  $x_{\mathbf{i}'} \otimes x_{\mathbf{i}''}$  are  $Q$ -linearly independent, by collecting the coefficient of each  $x_{\mathbf{i}'} \otimes x_{\mathbf{i}''}$  we obtain (3.9). The theorem is proved.  $\square$

Dualizing the assertions of Theorem 3.10, we obtain the following result.

**Proposition 3.12.** *In the assumption of Theorem 3.10, let  $\{x_w, w \in W\}$  (resp.  $\{\widehat{x}_{\widehat{w}}, \widehat{w} \in \widehat{W}\}$ ) be a  $Q$ -linear (resp.  $\widehat{Q}$ -linear) basis of  $Q_W$  (resp. of  $\widehat{Q}_{\widehat{W}}$ ) such that for all  $\widehat{w} \in \widehat{W}$ :*

$$\varphi(\widehat{x}_{\widehat{w}}) = \begin{cases} x_w & \text{if } \widehat{w} \in \widehat{W}_w \\ 0 & \text{if } \widehat{w} \notin \widehat{W}_w \end{cases}$$

where  $\widehat{W}_w \subset \widehat{W}$ ,  $w \in W$  is a finite subset of  $\widehat{W}$ . Then:

(a) For all  $u, v, w \in W$  and each  $\widehat{w} \in \widehat{W}_w$  one has:

$$(3.11) \quad p_{u,v}^w = \sum_{(\widehat{u}, \widehat{v}) \in \widehat{W}_u \times \widehat{W}_v} \varphi(\widehat{p}_{\widehat{u}, \widehat{v}}^{\widehat{w}}) .$$

(b) Assume additionally that  $f : Q \rightarrow Q'$  and  $\widehat{\varphi} : Q' \rightarrow \widehat{Q}'$  are homomorphisms of commutative  $\mathbb{k}$ -algebras such that:

- the set  $\{w \in W : f(p_{u,v}^w) \neq 0\}$  is finite for all  $u, v \in W$ ;
- the set  $\{\widehat{w} \in \widehat{W} : \widehat{f}(\widehat{p}_{\widehat{u}, \widehat{v}}^{\widehat{w}}) \neq 0\}$  is finite for all  $\widehat{u}, \widehat{v} \in \widehat{W}$ , where  $\widehat{f} = \widehat{\varphi} \circ f \circ \varphi$ .

Then, in the notation of Proposition 3.3, the association

$$\sigma_w \mapsto \sum_{\widehat{w} \in \widehat{W}_w} \widehat{\sigma}_{\widehat{w}}$$

defines a homomorphism of  $\mathbb{k}$ -algebras  $\varphi^* : \mathcal{A}_f \rightarrow \widehat{\mathcal{A}}_{\widehat{f}}$  such that  $\varphi^*|_{Q'} = \widehat{\varphi}$ .

*Proof.* Prove (a) Indeed, as in the proof of (3.9), applying  $\varphi$  to the expansion (3.2) for  $\widehat{Q}_{\widehat{W}}$  and using commutativity of (3.7), we obtain (3.11).

Prove (b) now. Indeed,

$$\begin{aligned} \varphi^*(\sigma_u \sigma_v) &= \varphi^*(\sigma_u \sigma_v) = \sum_{w \in W} \varphi^*(f(p_{u,v}^w) \sigma_w) = \sum_{w \in W} (\widehat{\varphi} \circ f)(p_{u,v}^w) \varphi^*(\sigma_w) \\ &= \sum_{w \in W, \widehat{w} \in \widehat{W}_w} (\widehat{\varphi} \circ f)(p_{u,v}^w) \widehat{\sigma}_{\widehat{w}} = \sum_{\substack{w \in W, \widehat{w} \in \widehat{W}_w, \\ (\widehat{u}, \widehat{v}) \in \widehat{W}_u \times \widehat{W}_v}} \widehat{f}(p_{\widehat{u}, \widehat{v}}^{\widehat{w}}) \widehat{\sigma}_{\widehat{w}} = \sum_{(\widehat{u}, \widehat{v}) \in \widehat{W}_u \times \widehat{W}_v} \widehat{\sigma}_{\widehat{u}} \widehat{\sigma}_{\widehat{v}} = \varphi^*(\sigma_u) \varphi^*(\sigma_v) . \end{aligned}$$

This proves (b).

The proposition is proved.  $\square$

Now we will compute all relative Littlewood-Richardson coefficients for our main class of  $S$ -tame sets  $X = \{x_i, i \in I\}$ , where

$$(3.12) \quad x_i = \alpha_i^{-1}(s_i - 1)$$

where  $\alpha_i$  are some invertible elements of  $Q$  and  $s_i \in W \setminus \{1\}$ . We sometimes refer to the elements  $x_i$  as *Demazure elements*.

**Corollary 3.13.** *For any  $S = \{s_i, i \in I\}$  the Demazure elements  $x_i$ ,  $i \in I$  and their monomials  $x_{\mathbf{i}}$ ,  $\mathbf{i} \in I^m$  satisfy:*

$$\delta(x_{\mathbf{i}}) = \sum_{\mathbf{i}', \mathbf{i}''} p_{\mathbf{i}', \mathbf{i}''}^{\mathbf{i}} x_{\mathbf{i}'} \otimes x_{\mathbf{i}''}$$

where the summation is over all pairs of subsequences  $(\mathbf{i}', \mathbf{i}'')$  of  $\mathbf{i}$  and the relative Littlewood-Richardson coefficients  $p_{\mathbf{i}', \mathbf{i}''}^{\mathbf{i}}$  are determined recursively by  $p_{\emptyset, \emptyset}^{\emptyset} = 1$  and:

$$(3.13) \quad p_{\mathbf{i}', \mathbf{i}''}^{\mathbf{i}} = x_{i_1}(p_{\mathbf{i}', \mathbf{i}''}^{\tilde{\mathbf{i}}}) + \delta_{i_1, i'_1} s_{i_1}(p_{\mathbf{i}', \mathbf{i}''}^{\tilde{\mathbf{i}}}) + \delta_{i_1, i''_1} s_{i_1}(p_{\mathbf{i}', \mathbf{i}''}^{\tilde{\mathbf{i}}}) + \delta_{i_1, i'_1} \delta_{i_1, i''_1} \alpha_{i_1} s_{i_1}(p_{\mathbf{i}', \mathbf{i}''}^{\tilde{\mathbf{i}}})$$

if  $m \geq 1$ , where  $\tilde{\mathbf{i}}$  stands for a sequence obtained from  $\mathbf{i}$  by deleting the first entry  $i_1$ .

*Proof.* Note that for the Demazure elements  $x_i$ , in the notation of (3.4), we have  $r_{ij} = \delta_{ij} \alpha_i^{-1}$ ,  $q_{ij} = \delta_{ij} \alpha_i$  for  $i, j \in I$ . Then the recursion (3.6) becomes (3.13). Finally, it follows from (3.13) (by induction in  $m$ ) that  $p_{\mathbf{i}', \mathbf{i}''}^{\mathbf{i}} = 0$  if either  $\mathbf{i}'$  or  $\mathbf{i}''$  is not a sub-sequence of  $\mathbf{i}$ .  $\square$

**Proposition 3.14.** *For any  $S = \{s_i, i \in I\}$ , a Demazure  $S$ -tame set  $X = \{x_i, i \in I\} \subset Q_W$  and any subalgebra  $R \subset Q$  such that all  $p_{\mathbf{i}', \mathbf{i}''}^{\mathbf{i}} \in R$  one has:*

(a) *There exists a commutative  $R$ -algebra  $\mathcal{A}_{X,R}$  with the basis  $\{\sigma_{\mathbf{i}}\}$ , where  $\mathbf{i}$  runs over all sequences in  $I^m$ ,  $m \geq 0$  such that:*

$$\sigma_{\mathbf{i}'} \sigma_{\mathbf{i}''} = \sum_{\mathbf{i}} p_{\mathbf{i}', \mathbf{i}''}^{\mathbf{i}} \sigma_{\mathbf{i}}$$

for any sequences  $\mathbf{i}' \in I^{m'}$  and  $\mathbf{i}'' \in I^{m''}$ , where the summation over all sequences  $\mathbf{i} \in I^m$ ,  $m \geq 0$  containing  $\mathbf{i}'$  and  $\mathbf{i}''$  as sub-sequences and such that  $m \leq m' + m''$ .

(b) *For each given  $\mathbf{i} \in I^m$  there exists an associative algebra  $\mathcal{A}_{X, \mathbf{i}, R}$  with the basis  $\{\sigma_{\mathbf{j}}^{(\mathbf{i})}\}$ , where  $\mathbf{j}$  runs over all subsequences of  $\mathbf{i}$  such that:*

$$\sigma_{\mathbf{j}'}^{(\mathbf{i})} \sigma_{\mathbf{j}''}^{(\mathbf{i})} = \sum_{\mathbf{j}} p_{\mathbf{j}', \mathbf{j}''}^{\mathbf{j}} \sigma_{\mathbf{j}}^{(\mathbf{i})}$$

for any subsequences  $\mathbf{j}', \mathbf{j}''$  of  $\mathbf{i}$ , where the summation over all subsequences  $\mathbf{j}$  of  $\mathbf{i}$ .

(c) *For each  $\mathbf{i} \in I^m$ ,  $m \geq 0$ , the association*

$$(3.14) \quad \sigma_{\mathbf{j}} \mapsto \begin{cases} \sigma_{\mathbf{j}}^{(\mathbf{i})} & \text{if } \mathbf{j} \text{ is a subsequence of } \mathbf{i} \\ 0 & \text{otherwise} \end{cases}$$

defines a surjective homomorphism of  $R$ -algebras  $\pi_{\mathbf{i}} : \mathcal{A}_{X,R} \rightarrow \mathcal{A}_{X, \mathbf{i}, R}$ .

*Proof.* Let now  $\widehat{W}$  be the free monoid generated by  $S = \{s_i, i \in I\}$ . Then the algebra  $\mathcal{A}_{X,R}$  is dual (over  $R$ ) of the coalgebra  $Q_{\widehat{W}}$ , i.e., is obtained by combining Lemma 3.4 (with  $Q = Q'$ ,  $f = id_Q$ ) and Theorem 3.10(b). The commutativity of  $\mathcal{A}_{X,R}$  follows from the symmetry

$$p_{\mathbf{i}'', \mathbf{i}'}^{\mathbf{i}} = p_{\mathbf{i}', \mathbf{i}''}^{\mathbf{i}},$$

which directly follows from the recursive definition (3.6). This proves (a).

Prove (b). Denote by  $\mathcal{C}_{\mathbf{i}} \subset Q_{\widehat{W}}$  the  $Q$ -linear span of all  $x_{\mathbf{j}}$  where  $\mathbf{j}$  runs over all subsequences of  $\mathbf{i}$ . It follows from Corollary 3.13 that  $\mathcal{C}_{\mathbf{i}}$  is closed under the coproduct  $\delta : Q_{\widehat{W}} \rightarrow Q_{\widehat{W}} \underset{Q}{\otimes} Q_{\widehat{W}}$  and thus  $\mathcal{C}_{\mathbf{i}}$  is a coalgebra in the category of  $Q$ -modules. Then applying Lemma 3.4 once again, we finish the proof of (b).

Prove (c) Indeed, the natural inclusion  $\mathcal{C}_{\mathbf{i}} \hookrightarrow Q_{\widehat{W}}$  is a homomorphism of coalgebras in the category of  $Q$ -modules. Its dual is a homomorphism of algebras given by (3.14). This proves (c)

The proposition is proved.  $\square$

We say that an index  $i_k$  of  $\mathbf{i} = (i_1, \dots, i_m) \in I^m$  is *repetition-free* if  $i_\ell \neq i_k$  for all  $\ell \in [m] \setminus \{k\}$ . And we say that  $\mathbf{i}$  is repetition-free if each  $i_k$ ,  $k \in [m]$  is repetition-free, i.e., all indices  $i_1, \dots, i_m$  are distinct (equivalently,  $|\{\mathbf{i}\}| = m$ , where  $\{\mathbf{i}\} = \{i_1, \dots, i_m\} \subset I$  denotes the underlying set).

For any subsequences  $\mathbf{i}'$ ,  $\mathbf{i}''$  of  $\mathbf{i}$  and a repetition-free index  $i$  of  $\mathbf{i}$  we define the  $f_i := f_i(\mathbf{i}, \mathbf{i}', \mathbf{i}'') \in Q_W$  by

$$(3.15) \quad f_i = \begin{cases} \alpha_i s_i & \text{if } i \in \{\mathbf{i}'\} \cap \{\mathbf{i}''\} \\ x_i & \text{if } i \notin \{\mathbf{i}'\} \cup \{\mathbf{i}''\} \\ s_i & \text{otherwise} \end{cases}$$

The following result computes all relative Littlewood-Richardson coefficients in the repetition-free case.

**Proposition 3.15.** *Assume that the indices  $i_1, i_2, \dots, i_k$  of  $\mathbf{i}$  are repetition-free. Then for any subsequences  $\mathbf{i}'$ ,  $\mathbf{i}''$  of  $\mathbf{i}$  we have:*

$$(3.16) \quad p_{\mathbf{i}', \mathbf{i}''}^{\mathbf{i}} = f_{i_1} f_{i_2} \cdots f_{i_k} (p_{\mathbf{i}' \setminus \{i_1, \dots, i_k\}, \mathbf{i}'' \setminus \{i_1, \dots, i_k\}}^{(i_{k+1}, \dots, i_m)})$$

*In particular, if  $\mathbf{i}$  is repetition-free, then  $p_{\mathbf{i}', \mathbf{i}''}^{\mathbf{i}}$  depends only on  $\mathbf{i}$ ,  $\{\mathbf{i}'\} \cap \{\mathbf{i}''\}$ , and  $\{\mathbf{i}'\} \cup \{\mathbf{i}''\}$ .*

*Proof.* If the index  $i_1$  is repetition-free, the recursion (3.13) drastically simplifies:

$$(3.17) \quad p_{\mathbf{i}', \mathbf{i}''}^{\mathbf{i}} = \begin{cases} \alpha_{i_1} s_{i_1} (p_{\mathbf{i}' \setminus \{i_1\}, \mathbf{i}'' \setminus \{i_1\}}^{\mathbf{i} \setminus \{i_1\}}) & \text{if } i'_1 = i''_1 = i_1 \\ s_{i_1} (p_{\mathbf{i}' \setminus \{i_1\}, \mathbf{i}''}^{\mathbf{i} \setminus \{i_1\}}) & \text{if } i'_1 = i_1 \neq i''_1 \\ s_{i_1} (p_{\mathbf{i}', \mathbf{i}'' \setminus \{i_1\}}^{\mathbf{i} \setminus \{i_1\}}) & \text{if } i''_1 = i_1 \neq i'_1 \\ x_{i_1} (p_{\mathbf{i}', \mathbf{i}''}^{\mathbf{i} \setminus \{i_1\}}) & \text{if } i_1 \notin \{i'_1, i''_1\} \end{cases} = f_{i_1} (p_{\mathbf{i}' \setminus \{i_1\}, \mathbf{i}'' \setminus \{i_1\}}^{(i_2, \dots, i_m)})$$

because in each of the cases in (3.13), all non-leading terms are zero (for instance, if  $i'_1 = i''_1 = i_1$ , then neither  $\mathbf{i}'$  nor  $\mathbf{i}''$  is a sub-sequence of  $\mathbf{i} \setminus \{i_1\}$ ). This proves (3.16) by induction.

The proposition is proved.  $\square$

When  $\mathbf{i}$  has repetitions, we can reduce the computation of the relative Littlewood-Richardson coefficients to the repetition-free case by introducing a certain class of relative repetition-free coefficients  $p_{K', K''}^{\mathbf{i}, K}$  which we refer to as *generalized Bott-Samelson coefficients*.

**Definition 3.16.** For any  $S = \{s_i, i \in I\} \subset W$  and any sequence  $\mathbf{i} = (i_1, \dots, i_m) \in I^m$  let  $\widehat{W}_{\mathbf{i}}$  be the free monoid generated by  $\widehat{S} = \{\widehat{s}_k\}$ ,  $k \in [m]$  and let  $\varphi_{\mathbf{i}} : \widehat{W}_{\mathbf{i}} \rightarrow W$  be the homomorphism of monoids given by  $\widehat{s}_k \mapsto s_{i_k}$  for  $k \in [m]$ . This makes any  $W$ -module algebra  $Q$  into a  $\widehat{W}_{\mathbf{i}}$ -module algebra and thus the twisted group algebra. Then  $Q_{\widehat{W}_{\mathbf{i}}} = Q \rtimes \widehat{W}_{\mathbf{i}}$  is well-defined. Next, we fix the Demazure  $\widehat{S}$ -tame set  $\widehat{X}_{\mathbf{i}} = \{\widehat{x}_k = \frac{1}{\alpha_{i_k}}(\widehat{s}_k - 1), i \in I, k \in [m]\}$ .

Then for any subsets  $K, K', K'' \subset [m]$  we set

$$p_{K', K''}^{\mathbf{i}, K} := \widehat{p}_{\mathbf{k}', \mathbf{k}''}^{\mathbf{k}}$$

where the right hand side is the relative coefficient for the twisted group algebra  $Q_{\widehat{W}_{\mathbf{i}}}$  with respect to  $\widehat{X}_{\mathbf{i}}$  and  $\mathbf{k} \in [m]^{|K|}$ ,  $\mathbf{k}' \in [m]^{|K'|}$ ,  $\mathbf{k}'' \in [m]^{|K''|}$  are the sequences naturally obtained from the sets  $K, K', K''$  respectively.

By definition,

$$(3.18) \quad p_{K', K''}^{\mathbf{i}, K} = p_{\varphi(K'), \varphi(K'')}^{\mathbf{i}_K, [|K|]}$$

for any  $K, K', K''$ , where  $\mathbf{i}_K$  is as in Definition 2.1 and  $\varphi : K \xrightarrow{\sim} \{1, \dots, |K|\}$  is the natural order-preserving bijection.

The following is a direct corollary of Theorem 3.10.

**Corollary 3.17.** For any  $\mathbf{i} \in I^m$  and any subsequences  $\mathbf{i}'$  and  $\mathbf{i}''$  of  $\mathbf{i}$  one has

$$(3.19) \quad p_{\mathbf{i}', \mathbf{i}''}^{\mathbf{i}} = \sum p_{K', K''}^{\mathbf{i}, [m]}$$

where the summation is over all pairs  $K', K'' \subset [m]$  such that  $\mathbf{i}_{K'} = \mathbf{i}'$ ,  $\mathbf{i}_{K''} = \mathbf{i}''$ . In particular, if  $\mathbf{i}$  is repetition-free, then:

$$(3.20) \quad p_{\mathbf{i}', \mathbf{i}''}^{\mathbf{i}} = p_{K', K''}^{\mathbf{i}, [m]}$$

where  $K' = \{\mathbf{i}'\}$ ,  $K'' = \{\mathbf{i}''\}$  (in the notation of (3.15)).

Since (3.19) is a copy of (2.13), this justifies the name *generalized Bott-Samelson coefficients* for  $p_{K', K''}^{\mathbf{i}, K}$ . We will make the analogy precise in the following result that generalizes [56, Proposition 4].

**Theorem 3.18.** Let  $S = \{s_i, i \in I\} \subset W$  and  $X = \{x_i = \frac{1}{\alpha_i}(s_i - 1), i \in I\} \subset Q_W$  be a Demazure  $S$ -tame set. Then for any sequence  $\mathbf{i} = (i_1, \dots, i_m) \in I^m$ , and any  $W$ -invariant subalgebra  $R \subset Q$  such that  $x_{i_k}(R) \subset R$  and  $\alpha_{i_k} \in R$  for  $k \in [m]$  one has:

(a) There exists a commutative  $R$ -algebra  $\mathcal{BS}_{X, \mathbf{i}, R}$  with the basis  $\{\sigma_K, K \subset [m]\}$  such that

$$(3.21) \quad \sigma_{K'} \sigma_{K''} = \sum_{K \subset [m]} p_{K', K''}^{\mathbf{i}, K} \sigma_K$$

for any subsets  $K', K'' \subset [m]$ , where the summation over all  $K \subset [m]$ , such that  $K' \cup K'' \subset K$  and  $|K| \leq |K'| + |K''|$ .

(b) For each  $K \subset [m]$  one has in  $\mathcal{BS}_{X,\mathbf{i},R}$ :

$$\sigma_K = \prod_{k \in K} \sigma_k .$$

(c) The algebra  $\mathcal{BS}_{X,\mathbf{i},R}$  is generated by  $\sigma_k := \sigma_{\{k\}}$ ,  $k \in [m]$  subject to the relations:

$$(3.22) \quad \sigma_k^2 = \alpha_{i_k} \sigma_k + \sum_{\ell < k} x_{i_k}(\alpha_{i_\ell}) \sigma_\ell \sigma_k$$

for  $k \in [m]$ .

(d) For each  $\mathbf{i} \in I^m$  the association

$$(3.23) \quad \sigma_{\mathbf{j}}^{(\mathbf{i})} \mapsto \sum_{\substack{K \subset [m]: \\ \mathbf{i}_K = \mathbf{j}}} \sigma_K$$

defines an injective homomorphism of  $R$ -algebras  $\pi_{\mathbf{i}} : \mathcal{A}_{X,\mathbf{i},R} \hookrightarrow \mathcal{BS}_{X,\mathbf{i},R}$ .

*Proof.* First note that the recursion (3.13) guarantees that the algebra  $R$  contains all  $p_{\mathbf{i}',\mathbf{i}''}^{\mathbf{i},K}$  and all  $p_{K',K''}^{\mathbf{i},K}$ .

Prove (a) now. In the notation of Proposition 3.14 define  $\mathcal{BS}_{X,\mathbf{i},R} := \mathcal{A}_{\widehat{X}_{\mathbf{i}},(1,2,\dots,m),R}$  and abbreviate

$$\sigma_K := \sigma_{\mathbf{j}}^{(1,2,\dots,m)} ,$$

for all  $K \subset J$  where  $\mathbf{j}$  is the only subsequence of  $(1,2,\dots,m)$  such that  $\{\mathbf{j}\} = K$ . Here the algebra  $\mathcal{A}_{\widehat{X}_{\mathbf{i}},(1,2,\dots,m),R}$  is associated to  $Q_{\widehat{W}_{\mathbf{i}}}$  with  $\widehat{W}_{\mathbf{i}}$  and  $\widehat{X}_{\mathbf{i}}$  as in Definition 3.16. Clearly, (3.21) holds in  $\mathcal{BS}_{X,\mathbf{i},R}$ . This proves (a).

Prove (b). It suffices to prove that  $\sigma_k \sigma_K = \sigma_{\{k\} \cup K}$  whenever  $k$  is less than the minimal element of  $K$ , or, equivalently,

$$(3.24) \quad p_{\{k\},K}^{\mathbf{i},\{k\} \cup K} = 1 .$$

Indeed, using (3.17), we see that

$$p_{\{k\},K}^{\mathbf{i},\{k\} \cup K} = \widehat{s}_1(p_{(k),(k_1,\dots,k_\ell)}^{(k,k_1,\dots,k_\ell)})$$

where  $K = \{k_1 < k_2 < \dots < k_\ell\}$ . Taking into the account that  $p_{\emptyset,\mathbf{k}}^{\mathbf{k}} = 1$  for all sequences  $\mathbf{k}$  (this easily follows by induction from (3.13)), we finish the proof of (3.24). Part (b) is proved.

Prove (c). Indeed, it follows from Definition 3.16, (3.20) and (3.17) that for any  $K = \{k_1 < k_2\} \subset [m]$  and  $k \in [m]$  one has

$$p_{\{k\},\{k\}}^{\mathbf{i},\{k_1,k_2\}} = \delta_{k_1,k} p_{(k),(k)}^{(k,k_2)} + \delta_{k_2,k} p_{(k),(k)}^{(k_1,k)} = \delta_{k_1,k} \widehat{s}_k(p_{\emptyset,\emptyset}^{(k_2)}) + \delta_{k_2,k} \widehat{x}_{k_1}(p_{(k),(k)}^{(k)}) = \delta_{k_2,k} x_{i_{k_1}}(\alpha_{i_k}) .$$

Taking into account that  $p_{\{k\},\{k\}}^{\mathbf{i},\{k\}} = p_{(i_k),(i_k)}^{(i_k)} = \alpha_{i_k}$ , we obtain:

$$\begin{aligned} \sigma_k^2 &= \sum_{\substack{K \subset [m] \\ k \in K, |K| \leq 2}} p_{\{k\},\{k\}}^{\mathbf{i},K} \sigma_K = \sum_{k' \in [m]} p_{\{k\},\{k\}}^{\mathbf{i},\{k,\ell\}} \sigma_{\{k,\ell\}} = \alpha_{i_k} \sigma_k + \sum_{\ell \neq k} p_{\{k\},\{k\}}^{\mathbf{i},\{k,\ell\}} \sigma_{\{k,\ell\}} \\ &= \alpha_{i_k} \sigma_k + \sum_{\ell \neq k} \delta_{\max(k,\ell),k} \sigma_{\{k,\ell\}} = \alpha_{i_k} \sigma_k + \sum_{\ell < k} \sigma_\ell \sigma_k . \end{aligned}$$

This proves (3.22). It remains to prove that the relations (3.22) are defining. This follows from the following result.

**Proposition 3.19.** *Let  $\mathcal{A}$  be a commutative  $R$ -algebra generated by  $\sigma_k$ ,  $k \in [m]$  subject to the relations:*

$$(3.25) \quad \sigma_k^2 = c_k \sigma_k + \sum_{\ell < k} c_{\ell,k} \sigma_\ell \sigma_k$$

for all  $k \in [m]$ , where all  $c_k$  and  $c_{\ell,k}$  belong to  $R$ . Then the set of square-free monomials in  $\sigma_1, \dots, \sigma_m$  is a free  $R$ -linear basis of  $\mathcal{A}$  (hence  $\dim_R \mathcal{A} = 2^m$ ).

*Proof.* Define the valuation  $\nu : R[\sigma_1, \dots, \sigma_m] \setminus \{0\} \rightarrow \mathbb{Z}_{\geq 0}^m$  by

$$\nu\left(\sum c_{\mathbf{r}} \sigma^{\mathbf{r}}\right) = \max\{\mathbf{r} \mid \mathbf{r} \in \mathbb{Z}_{\geq 0}^m : c_{\mathbf{r}} \neq 0\}$$

where we abbreviated  $\sigma^{\mathbf{r}} = \sigma_1^{r_1} \cdots \sigma_m^{r_m}$  and the maximum is taken with respect to the the *inverse lexicographic* ordering on  $\mathbb{Z}_{\geq 0}^m$  (i.e.,  $\mathbf{r} < \mathbf{r}'$  if there exists  $k \in [m]$  such that  $r_k < r'_k$  and  $r_\ell = r'_\ell$  for all  $\ell > k$ ).

For each  $p = \sum c_{\mathbf{r}} \sigma^{\mathbf{r}} \in R[\sigma_1, \dots, \sigma_m] \setminus \{0\}$  define the the leading coefficient  $c(p)$  by

$$c(p) := c_{\nu(p)} .$$

We say that a subset  $B$  of  $R[\sigma_1, \dots, \sigma_m] \setminus \{0\}$  is *triangular* if

$$\nu(B) = \mathbb{Z}_{\geq 0}^m, c(B) = \{1\}$$

The following fact is obvious.

**Lemma 3.20.** *Each triangular subset  $B$  of  $R[\sigma_1, \dots, \sigma_m] \setminus \{0\}$  is a free  $R$ -linear basis of  $R[\sigma_1, \dots, \sigma_m]$  and the transition matrix between  $B$  and the standard monomial basis  $\sigma^{\mathbb{Z}_{\geq 0}^m}$  is unitriangular with respect to the inverse lexicographic order.*

In the notation of (3.25) define  $M_k \in R[\sigma_1, \dots, \sigma_m] \setminus \{0\}$ ,  $k \in [m]$  by

$$M_k = \sigma_k^2 - c_k \sigma_k - \sum_{\ell < k} c_{\ell,k} \sigma_\ell \sigma_k .$$

Clearly, the quotient algebra  $R[\sigma_1, \dots, \sigma_m]/J_m$ , where  $J_m$  is the ideal generated by all  $M_k$ ,  $k \in [m]$  is isomorphic to  $\mathcal{A}$ .

It is also clear that  $\nu(M_k) = 2e_k$  for  $k = 1, \dots, m$ , where  $e_1, \dots, e_m$  is the standard basis of  $\mathbb{Z}_{\geq 0}^m$ .

Denote by  $\mathcal{M}$  the  $R$ -subalgebra of  $R[\sigma_1, \dots, \sigma_m]$  generated by  $M_1, \dots, M_m$ . For each  $\mathbf{t} \in \mathbb{Z}_{\geq 0}^m$  define a monomial  $M^{\mathbf{t}} \in \mathcal{M}$  by:

$$M^{\mathbf{t}} := M_1^{t_1} \cdots M_m^{t_m} .$$

Denote by  $B_0$  the set of all square-free monomials in  $R[\sigma_1, \dots, \sigma_m]$ , i.e., all  $\sigma^{\mathbf{r}}$  with  $\max_{k \in [m]}(r_k) \leq 1$ . The following fact is obvious.

**Lemma 3.21.** *The set*

$$B = \sqcup_{\mathbf{t} \in \mathbb{Z}_{\geq 0}^m} M^{\mathbf{t}} \cdot B_0$$

*is triangular. In particular,  $\mathcal{M} \cong R[x_1, \dots, x_m]$  and  $R[\sigma_1, \dots, \sigma_m]$  is a free  $\mathcal{M}$ -module with the free  $\mathcal{M}$ -linear basis  $B_0$  of square-free monomials.*



This implies that  $\sqcup_{t \in \mathbb{Z}_{\geq 0}^m \setminus \{0\}} M^t \cdot B_0$  is a free  $R$ -linear basis in the ideal  $J_m$ . Hence the restriction to  $B_0$  of the quotient map  $R[\sigma_1, \dots, \sigma_m] \rightarrow R[\sigma_1, \dots, \sigma_m]/J_m = \mathcal{A}$  is injective and the image of  $B_0$  is a free  $R$ -linear basis of  $\mathcal{A}$ .

The proposition is proved.  $\square$

This finishes the proof of part (c).

Prove (d) now. Indeed, Theorem 3.10(b) guarantees that (in the notation of the proof of 3.14(b)), the homomorphism of monoids  $\widehat{W}_{\mathbf{i}} \rightarrow W$  given by  $\widehat{s}_k \mapsto s_{i_k}$  extends to we have a surjective homomorphism of coalgebras  $\varphi_{\mathbf{i}} : \widehat{\mathcal{C}}_{(1,2,\dots,m)} \twoheadrightarrow \mathcal{C}_{\mathbf{i}}$ . Therefore, dualizing this surjective homomorphism over  $R$ , we obtain an injective homomorphism  $\pi_{\mathbf{i}} : \mathcal{A}_{X,\mathbf{i},R} \hookrightarrow \mathcal{BS}_{X,\mathbf{i},R}$  given by (3.23). Part (d) is proved.

Therefore, Theorem 3.18 is proved.  $\square$

**Remark 3.22.** In fact, the homomorphism (3.23) verifies (3.19).

#### 4. GENERALIZED NIL HECKE ALGEBRAS AND PROOF OF THEOREM 2.14

Let  $I$  be a finite set of indices. We say that an  $I \times I$  matrix  $A = (a_{i,j})$  over  $\mathbb{k}$  is *quasi-Cartan* if all  $a_{ii} = 2$  and  $a_{ij}a_{ji} = 0$  implies  $a_{ij} = a_{ji} = 0$ . Let  $V$  be a  $\mathbb{k}$  vector space with basis  $\{\alpha_i, i \in I\}$ .

**Definition 4.1.** We say that a monoid generated by  $S = \{s_i, i \in I\}$  is a *Coxeter semigroup* if  $W$  is subject to the relations

$$(s_i s_j)^{n_{ij}} = 1$$

for all  $i, j \in I$ , where  $n_{ii} \in \{0, 2\}$ ,  $n_{ij} = n_{ji} \in \{0\} \cup \mathbb{Z}_{\geq 2}$  for all  $i, j$ ; and: if  $n_{ii} = 0$  for some  $i$ , then  $n_{ij} = 0$  for all  $j$ .

**Remark 4.2.** In fact, the term ‘‘Coxeter monoid’’ has been used by several authors to denote a different object (see e.g., [18, 47, 52, 53]) that is never a group. At the same time, any Coxeter semigroup with all  $n_{ii} = 2$  is a Coxeter group.

Note that the free monoid generated by  $S$  is a Coxeter semigroup, moreover it is an initial object in the category of Coxeter semigroups generated by  $S$ .

The following fact is obvious.

**Lemma 4.3.** *Let  $W$  be a Coxeter semigroup generated by  $S = \{s_i, i \in I\}$  and let  $I_0 = \{i \in I : n_{ii} = 0\}$ . Then*

- (a) *The sub-monoid of  $W$  generated by all  $s_i, i \in I_0$  is a free monoid  $M_0$*
- (b) *The sub-monoid of  $W$  generated by all  $s_i, i \in I \setminus I_0$  is a Coxeter group  $W_0$ .*
- (c) *One has  $W = W_0 \star M_0$ , where  $\star$  stands for the free product of monoids.*

**Definition 4.4.** Similarly to Definition 2.1, given a Coxeter semigroup, we say that a sequence  $\mathbf{i} = (i_1, \dots, i_m) \in I^m$  is *reduced* if the element  $w = w_{\mathbf{i}} := s_{i_1} \cdots s_{i_m} \in W$  is shortest possible in  $W$  and define its *Coxeter length*  $\ell(w) := m$ . Given  $w \in W$ , denote by  $R(w)$  the set of all *reduced words* of  $w$ , i.e, all  $\mathbf{i} \in I^{\ell(w)}$  such that  $w_{\mathbf{i}} = w$ .

**Definition 4.5.** We say that a Coxeter semigroup  $W$  is *weakly compatible* with a quasi-Cartan matrix  $A$  if for each  $i \neq j$  we have:

$$n_{ij} \geq 2 \text{ implies that } a_{ij}a_{ji} = \zeta_{ij} + \zeta_{ij}^{-1} + 2$$

for some  $n_{ij}$ -th root of unity  $\zeta_{ij} \in \mathbb{k}^\times$ .

The following result is obvious.

**Lemma 4.6.** *Let  $V$  be a  $\mathbb{k}$  vector space with basis  $\{\alpha_i, i \in I\}$  and let  $A = (a_{ij})$  be an  $I \times I$  quasi-Cartan matrix weakly compatible with a Coxeter semigroup  $W$  generated by  $S = \{s_i, i \in I\}$ . Then the association*

$$s_i(\alpha_j) = \alpha_i - a_{ij}\alpha_j$$

*defines an action of  $W$  on  $V$ .*

Throughout the section we fix a Coxeter semigroup  $W = \langle s_i, i \in I \rangle$  and a weakly compatible quasi-Cartan matrix  $A = (a_{ij})$  together with the action  $W \times V \rightarrow V$  prescribed by Lemma 4.6. Denote by  $Q = \text{Frac}(S(V))$  the field of fractions of the symmetric algebra  $S(V)$ .

Clearly,  $Q$  is a module algebra over the group (or, rather, monoidal) algebra  $\mathbb{k}W$  so one has a twisted group algebra  $Q_W := Q \rtimes \mathbb{k}W$  and the coaction

$$(4.1) \quad \delta : Q_W \rightarrow Q_W \bigotimes_Q Q_W$$

given by Proposition 3.1.

For any  $i \in I$ , define a Demazure element  $x_i \in Q_W$  by:

$$x_i := \frac{1}{\alpha_i}(s_i - 1)$$

and denote by  $\mathcal{H}_A(W)$  the subalgebra of  $Q_W$  generated by all  $x_i, i \in I$ . Following Kostant and Kumar ([31]), we refer to it as a *generalized nil Hecke algebra*.

**Theorem 4.7.** *Assume that a Coxeter semigroup  $W$  and a quasi-Cartan matrix  $A$  are weakly compatible and  $\sqrt{a_{ij}a_{ji}} \in \mathbb{k}$  whenever  $n_{ij}$  is odd. Then the generalized nil Hecke algebra  $\mathcal{H}_A(W)$  is subject to the following relations:*

$$(4.2) \quad x_i^2 = 0 \text{ iff } n_{ii} = 2; \quad \begin{cases} \underbrace{x_i x_j \cdots x_j}_{n_{ij}} = \underbrace{x_j x_i \cdots x_i}_{n_{ij}} & \text{if } n_{ij} \in 2\mathbb{Z}_{>0} \\ \underbrace{x_i x_j \cdots x_i}_{n_{ij}} = \sqrt{\frac{a_{ij}}{a_{ji}}} \underbrace{x_j x_i \cdots x_j}_{n_{ij}} & \text{if } n_{ij} \in 1 + 2\mathbb{Z}_{\geq 0} \end{cases}$$

*In particular, the monomials  $x_{\mathbf{i}} = x_{i_1} \cdots x_{i_m}$  satisfy:*

$$(4.3) \quad x_{\mathbf{i}} = 0$$

*for any non-reduced sequence  $\mathbf{i}$  and*

$$(4.4) \quad x_{\mathbf{i}} = d_{\mathbf{i}, \mathbf{i}'} x_{\mathbf{i}'}$$

*for any  $\mathbf{i}, \mathbf{i}' \in R(w)$ ,  $w \in W$ , where  $d_{\mathbf{i}, \mathbf{i}'} \in \mathbb{k}^\times$  is a product of  $\sqrt{\frac{a_{ij}}{a_{ji}}}$ .*

*Proof.* Indeed, if  $n_{ii} = 2$ , i.e.,  $s_i^2 = 1$ , then the relation  $x_i^2 = 0$  is obvious.

The remaining relations follow from the following rank 2 computation.

**Proposition 4.8.** *Assume that  $I = \{1, 2\}$ ,  $W = \langle s_1, s_2 \mid s_1^2 = s_2^2 = (s_1 s_2)^n = 1 \rangle$  is a dihedral group, and  $A = \begin{pmatrix} 2 & a_{12} \\ a_{21} & 2 \end{pmatrix}$  is a quasi-Cartan matrix over  $\mathbb{k}$  with  $a_{12}a_{21} = \zeta + \zeta^{-1} + 2$ , where  $\zeta \in \mathbb{k}$  is an  $n$ -th root of unity and  $\sqrt{a_{12}a_{21}} \in \mathbb{k}$  if  $n$  is odd. Then the generators of the generalized nil Hecke algebra  $\mathcal{H}_A(W)$  satisfy:*

$$(4.5) \quad \begin{cases} \underbrace{x_1 x_2 \cdots x_2}_n = \underbrace{x_2 x_1 \cdots x_1}_n & \text{if } n \text{ is even} \\ \underbrace{x_1 x_2 \cdots x_1}_n = \sqrt{\frac{a_{21}}{a_{12}}} \underbrace{x_2 x_1 \cdots x_2}_n & \text{if } n \text{ is odd} \end{cases}.$$

*Proof.* We will follow the proof of [31, Proposition 4.2]. Let  $V = \mathbb{k}\alpha_1 \oplus \mathbb{k}\alpha_2$  and  $\langle \cdot, \alpha_j^\vee \rangle$ ,  $j = 1, 2$ , be a linear function  $V \rightarrow \mathbb{k}$  given by  $\langle \alpha_i, \alpha_j^\vee \rangle = a_{ij}$ . Without loss of generality, by rescaling  $\alpha_i$  and  $\alpha_i^\vee$ ,  $i = 1, 2$ , we assume that  $a_{12} = a_{21}$  if  $n$  is odd.

The following result is obvious.

**Lemma 4.9.** *In the assumptions of Proposition 4.8 and that  $a_{12} = a_{21}$  if  $n$  is odd, we have:*

(a) *for any  $\alpha \in V$ :*

$$\alpha \underbrace{x_i x_j \cdots x_{i'}}_m = \underbrace{x_i x_j \cdots x_{i'}}_m \cdot w_m^{-1}(\alpha) - \langle \alpha, \alpha_i^\vee \rangle \cdot \underbrace{x_j x_i \cdots}_{m-1} - \langle w_{m-1}^{-1}(\alpha), \alpha_{i'}^\vee \rangle \cdot \underbrace{x_i x_j \cdots}_{m-1}$$

for  $m \in \mathbb{Z}_{>0}$  and  $\{i, j\} = \{1, 2\}$ , where  $i' = i$  if  $m$  is odd and  $i' = j$  if  $m$  is even, and

$$w_m = w_m^{(i)} = \underbrace{s_i s_j \cdots s_{i'}}_m.$$

(b)  $\underbrace{x_i x_j \cdots x_{i'}}_m \in c_m^{(i)} w_m^{(i)} + \sum_{w: \ell(w) < \ell(w_m^{(i)})} Q \cdot w$  for  $\{i, j\} = \{1, 2\}$ , where

$$c_k^{(i)} = \frac{1}{\alpha_i \cdot s_i(\alpha_j) \cdots w_{k-1}^{(i)}(\alpha_{i'})}.$$

(c) *The action of the longest element  $w_\circ := w_n^{(1)} = w_n^{(2)}$  on  $V$  and  $V^*$  is given by:*

$$w_\circ(\alpha_i) = \begin{cases} -\alpha_i & \text{if } n \text{ is even} \\ -\alpha_j & \text{if } n \text{ is odd} \end{cases}, \quad w_\circ(\alpha_i^\vee) = \begin{cases} -\alpha_i^\vee & \text{if } n \text{ is even} \\ -\alpha_j^\vee & \text{if } n \text{ is odd} \end{cases}$$

for  $\{i, j\} = \{1, 2\}$ , where  $\ell(w)$  is the Coxeter length of  $w$  (see Definition 4.4).

This implies that

$$\alpha \underbrace{x_1 x_2 \cdots x_2}_n - \underbrace{x_1 x_2 \cdots x_2}_n \cdot w_\circ^{-1}(\alpha) = \alpha \underbrace{x_2 x_1 \cdots x_1}_n - \underbrace{x_2 x_1 \cdots x_1}_n \cdot w_n^{-1}(\alpha)$$

for all  $\alpha \in V$ . Equivalently

$$(4.6) \quad \alpha \cdot \mathbf{D} = \mathbf{D} \cdot w_\circ^{-1}(\alpha)$$

for all  $\alpha \in V$ , where  $\mathbf{D} := \underbrace{x_1 x_2 \cdots}_n - \underbrace{x_2 x_1 \cdots}_n$ . Let us prove that  $\Delta = 0$ . First, parts (b) and (c) of Lemma 4.9 imply that  $c_n^{(1)} = c_n^{(2)}$ . Therefore,

$$\mathbf{D} = \sum_{w: \ell(w) < \ell(w_o)} c_w w$$

for some  $c_w \in Q = \mathbb{k}(\alpha_1, \alpha_2)$ . Hence, (4.6) becomes:

$$0 = \sum_{w: \ell(w) < \ell(w_o)} c_w \cdot (\alpha \cdot w - w \cdot w_o^{-1}(\alpha)) = \sum_{w: \ell(w) < \ell(w_o)} c_w \cdot (\alpha - w w_o^{-1}(\alpha)) \cdot w$$

for all  $\alpha \in V$ . This implies that all  $c_w = 0$  hence  $\mathbf{D} = 0$ .

The proposition is proved.  $\square$

This proves the relations (4.2) in  $\mathcal{H}_A(W)$  which immediately imply (4.3) and (4.4). Let us verify that the relations (4.2) are defining. For each  $w \in W$  let us choose a representative  $\mathbf{i}_w \in R(w)$ . Then

$$\sum_{w \in W} \mathbb{k} \cdot x_{\mathbf{i}_w} = \mathcal{H}_A(W)$$

by (4.3) and (4.4). Note that  $Q \cdot \mathcal{H}_A(W) = Q_W$  since  $X = \{x_i, i \in I\}$  is tame, and therefore

$$(4.7) \quad \sum_{w \in W} Q \cdot x_{\mathbf{i}_w} = Q_W.$$

It suffices to prove that this sum is direct, i.e.,  $\{x_{\mathbf{i}_w} \mid w \in W\}$  is a  $Q$ -basis of  $Q_W$ . Indeed, it is easy to see that  $Q_W$  is filtered by  $Q$ -submodules  $(Q_W)_{\leq m} = \bigoplus_{w: \ell(w) \leq m} Q \cdot w$

and that for any  $\mathbf{i} = (i_1, \dots, i_m) \in R(w)$  one has:

$$x_{\mathbf{i}} \equiv \frac{1}{\alpha_{i_1}} s_{i_1} \cdots \frac{1}{\alpha_{i_m}} s_{i_m} \pmod{(Q_W)_{\leq m-1}}$$

hence  $x_{\mathbf{i}_w} \in Q^\times w + (Q_W)_{\leq \ell(w)-1}$  for all  $w \in W$ . Therefore,  $\{x_{\mathbf{i}_w} \mid w \in W\}$  is a  $Q$ -basis of  $Q_W$  and Theorem 4.7 is proved.  $\square$

**Remark 4.10.** In fact, we can explicitly compute the expansion of elements  $\underbrace{x_i x_j \cdots x_{i'}}_k$

in Lemma 4.9(b) by generalizing the recursion [33, Equation (8)]. Namely, in the notation of Lemma 4.9, assume that  $a_{12} = a_{21} = t + t^{-1}$ . Then the coefficients  $d_w^{(i)} = d_w^{(i;m)} \in \mathbb{Z}[a_{12}, \alpha_1, \alpha_2]$  of the expansion:

$$\underbrace{x_i x_j \cdots}_m = c_m^{(i)} w_m^{(i)} + \sum_{w \in W: \ell(w) < m} d_w^{(i)} w$$

are given by  $\underbrace{d_{s_j s_i \dots}^{(i)}}_k = -\underbrace{d_{s_i s_j \dots}^{(i)}}_{k+1}$  for  $0 \leq k < m$  and:

$$\underbrace{d_{s_i s_j \dots}^{(i)}}_{m-2k} = \begin{bmatrix} m-1 \\ k \end{bmatrix}_t c_k^{(j)} \cdot c_{m-k}^{(i)}, \quad \underbrace{d_{s_i s_j \dots}^{(i)}}_{m-2k-1} = - \begin{bmatrix} m-1 \\ k \end{bmatrix}_t c_k^{(i)} \cdot c_{m-k-1}^{(j)}$$

for  $0 \leq k < \frac{m}{2}$ ,  $\{i, j\} = \{1, 2\}$ , where  $\begin{bmatrix} m-1 \\ k \end{bmatrix}_t \in \mathbb{Z}[a_{12}]$  are binomial polynomials in  $t$  (as in Remark 7.3).

**Definition 4.11.** We say that a Coxeter semigroup  $W$  and a weakly compatible quasi-Cartan matrix  $A$  are *compatible* if  $n_{ij} \in 2\mathbb{Z} + 1$  implies that  $a_{ij} = a_{ji}$ .

Clearly, for any  $I \times I$  quasi-Cartan matrix  $A$ , the free Coxeter group  $\widetilde{W} = \langle s_i, i \in I : s_i^2 = 1 \rangle$  and the free monoid on  $S = \{s_i, i \in I\}$  are both compatible with  $A$ .

Assume now that  $A$  and  $W$  are compatible, then  $d_{\mathbf{i}, \mathbf{i}'} = 1$  in (4.4) and for each  $w \in W$  there exists an element  $x_w \in \mathcal{H}_A(W)$  such that

$$x_w = x_{\mathbf{i}}$$

for all  $\mathbf{i} \in R(w)$ . Clearly, the collection  $\{x_w \mid w \in W\}$  is determined by  $x_{s_i} = x_i$  for  $i \in I$  and

$$x_u x_v = \begin{cases} x_{uv} & \text{if } \ell(uv) = \ell(u) + \ell(v) \\ 0 & \text{if } \ell(uv) < \ell(u) + \ell(v) \end{cases}.$$

The following is an immediate corollary from the proof of Theorem 4.7.

**Corollary 4.12.** Assume that a Coxeter semigroup  $W$  and a quasi-Cartan matrix  $A$  are compatible. Then the collection  $\{x_w \mid w \in W\}$  is a basis of the generalized nil Hecke algebra  $\mathcal{H}_A(W)$ .

In particular,  $B = \{x_w \mid w \in W\}$  is a left  $Q$ -basis of  $Q_{A,W}$  (in the notation of Section 3). This defines the Littlewood-Richardson coefficients  $p_{u,v}^w = p_{u,v}^w(A) \in Q$  for  $u, v, w \in W$  by (4.1) and the formula (3.2). Similarly, for each admissible (in the sense of Definition 2.1) sequence  $\mathbf{i} \in I^m$  and its subsequences  $\mathbf{i}'$  and  $\mathbf{i}''$  one defines the corresponding relative Littlewood-Richardson coefficient  $p_{\mathbf{i}', \mathbf{i}''}^{\mathbf{i}} = p_{\mathbf{i}', \mathbf{i}''}^{\mathbf{i}}(A)$ . If  $W$  is the free monoid (resp. the free Coxeter group) on  $S$ , then the assignment

$$(4.8) \quad \mathbf{i} = (i_1, \dots, i_m) \mapsto \widehat{w}_{\mathbf{i}} = s_{i_1} \cdots s_{i_m}$$

is a bijection between the set of all (resp. all admissible) sequences and  $W$ , e.g.,  $p_{\mathbf{i}', \mathbf{i}''}^{\mathbf{i}} = p_{\widehat{w}_{\mathbf{i}'}, \widehat{w}_{\mathbf{i}''}}^{\widehat{w}_{\mathbf{i}}}$ .

**Proposition 4.13.** For each quasi-Cartan matrix  $A$  and a weakly compatible Coxeter semigroup  $W$  one has:

(a) Each  $p_{\mathbf{i}', \mathbf{i}''}^{\mathbf{i}}$  belongs to  $S(V)$  and is homogeneous of degree  $m' + m'' - m$ , where  $m, m'$ , and  $m''$  are respectively the lengths of  $\mathbf{i}, \mathbf{i}'$ , and  $\mathbf{i}''$ .

(b) Assume that  $A$  and  $W$  are compatible. Then for each triple  $u, v, w \in W$  with  $\ell(u) + \ell(v) \geq \ell(w)$  and for each  $\mathbf{i} \in R(w)$  one has:

$$(4.9) \quad p_{u,v}^w = \sum_{\mathbf{i}' \in R(u), \mathbf{i}'' \in R(v)} p_{\mathbf{i}', \mathbf{i}''}^{\mathbf{i}}.$$

*Proof.* Part (a) directly follows from the recursion (3.6) and the following obvious fact.

**Lemma 4.14.** *Under the action of  $x_i = \frac{1}{\alpha_i}(s_i - 1)$  on  $Q = \text{Frac}(S(V))$  one has:*

$$x_i(\alpha_j) = -a_{ij}, x_i(fg) = x_i(f)g + s_i(f)x_i(g)$$

for all  $i, j \in I$ ,  $f, g \in Q$ . In particular,

$$x_i(S^k(V)) \subset S^{k-1}(V)$$

for each  $k \geq 0$ .

Prove (b). Indeed, in the notation of Theorem 3.10(a), let  $\widehat{W}$  be the free Coxeter group generated by  $\widehat{s}_i$ ,  $i \in I$  with the structural surjective homomorphism  $\varphi_- : \widehat{W} \rightarrow W$  given by  $\widehat{s}_i \mapsto s_i$ , which, together with the identity map  $Q \rightarrow Q$  extends to an algebra homomorphism  $\varphi : Q_{\widehat{W}} \rightarrow Q_W$  such that  $\varphi(\widehat{x}_{\widehat{\mathbf{i}}}) = x_w$  for all  $\mathbf{i} \in R(w)$  under the bijection (4.8), i.e.,  $\widehat{W}_w = R(w)$ . Finally, taking into account that  $p_{\widehat{w}_1, \widehat{w}_2}^{\widehat{\mathbf{i}}} = p_{\mathbf{i}', \mathbf{i}''}^{\mathbf{i}}$ , the identity (3.11) becomes (4.9). This proves (b).

The proposition is proved.  $\square$

As a corollary, we obtain a generalization of [31, Proposition 4.15].

**Corollary 4.15.** *Each  $p_{u,v}^w$  belongs to  $S(V)$  and is homogeneous of degree  $\ell(u) + \ell(v) - \ell(w)$  (e.g.,  $p_{u,v}^w = 0$  if  $\ell(u) + \ell(v) < \ell(w)$ ).*

Dualizing the above arguments, we obtain the following result.

**Proposition 4.16.** *For each quasi-Cartan matrix  $A$  and any compatible Coxeter semigroup  $W$  we have:*

(a) *There is a unique commutative  $S(V)$ -algebra  $\mathcal{A}_A(W)$  with the free  $S(V)$ -basis  $\{\sigma_w \mid w \in W\}$  and the following multiplication table:*

$$\sigma_u \sigma_v = \sum_{w \in W} p_{u,v}^w \sigma_w$$

for all  $u, v \in W$ .

(b) *There is a unique commutative  $S(V)$ -algebra  $\widehat{\mathcal{A}}_A$  with the free  $S(V)$ -basis  $\{\sigma_{\mathbf{i}}\}$  labeled by all sequences in  $I^m$ ,  $m \geq 0$  with following multiplication table:*

$$(4.10) \quad \sigma_{\mathbf{i}'} \sigma_{\mathbf{i}''} = \sum_{\mathbf{i}} p_{\mathbf{i}', \mathbf{i}''}^{\mathbf{i}} \sigma_{\mathbf{i}}$$

where the summation over all sequences  $\mathbf{i} \in I^m$ ,  $m \geq 0$  containing  $\mathbf{i}'$  and  $\mathbf{i}''$  as sub-sequences and such that  $m \leq m' + m''$ .

(c) One has an injective algebra homomorphism  $\mathcal{A}_A(W) \hookrightarrow \widehat{\mathcal{A}}_A$  via:

$$\sigma_w \mapsto \sum_{\mathbf{i} \in R(w)} \sigma_{\mathbf{i}} .$$

The following is a slight modification ( $Q$  is replaced with  $S(V)$ ) of Corollary 3.5.

**Corollary 4.17.** *Given a Coxeter semigroup  $W$  and a compatible quasi-Cartan matrix  $A$ , let  $\langle \cdot, \cdot \rangle : \mathcal{A}_A(W) \times S(V) \cdot \mathcal{H}_A(W) \rightarrow S(V)$  be the non-degenerate  $S(V)$ -bilinear pairing given by*

$$\langle p\sigma_u, qx_v \rangle = \delta_{u,v} \cdot pq$$

for all  $u, v \in W$ ,  $p, q \in S(V)$ . Then:

(a) The above pairing satisfies:

$$\langle ab, x \rangle = \langle a \otimes b, \delta(x) \rangle = \langle a, x_{(1)} \rangle \langle b, x_{(2)} \rangle$$

for all  $a, b \in \mathcal{A}_A(W)$ ,  $x \in S(V) \cdot \mathcal{H}_W$ , where  $\delta(x) = x_{(1)} \otimes x_{(2)}$  in the Sweedler notation.

(b) For each  $w \in W$  the assignment  $a \mapsto \langle a, w \rangle$ ,  $a \in \mathcal{A}_A(W)$  is an  $S(V)$ -algebra homomorphism

$$\varphi_w : \mathcal{A}_A(W) \rightarrow S(V) .$$

**Remark 4.18.** If  $A$  is a Cartan matrix and  $W$  is the Weyl group of a Kac-Moody group  $G$ , Kostant and Kumar proved that  $\varphi_w$  is a homomorphism of algebras (see [34, Section 11.1.4]). Sara Billey computed  $\varphi_w(\sigma_v)$  explicitly in [8, Theorem 4].

The algebras  $\mathcal{A}_A(W)$  are very important in Schubert Calculus due to the following fundamental result.

**Theorem 4.19.** ([34, Corollary 11.3.17]) *Let  $G$  be a complex semisimple or Kac-Moody group,  $T \subset B$  be respectively the Cartan and Borel subgroups of  $G$ ,  $W = \text{Norm}_G(T)/T$  be the Weyl group, and let  $A$  be the Cartan matrix of  $G$ . Then the assignment*

$$\sigma_w^T \mapsto \sigma_w$$

defines an isomorphism of  $S(V)$ -algebras  $H_T^*(G/B) \xrightarrow{\sim} \mathcal{A}_A(W)$ , where  $H_T^*(G/B)$  is the  $T$ -equivariant cohomology algebra (over  $S(V) = H_T^*(pt)$ ) of  $G/B$  and  $\sigma_w^T$ ,  $w \in W$  is the  $T$ -equivariant Schubert cocycle corresponding to  $w \in W$ . In particular, the cup product in  $H_T^*(G/B)$  is given by:

$$\sigma_u^T \cup \sigma_v^T = \sum_{w \in W} p_{u,v}^w \sigma_w^T .$$

Therefore, the cup product in the cohomology algebra  $H^*(G/B, \mathbb{C}) = \mathbb{C} \bigotimes_{S(V)} H_T^*(G/B)$

(where  $\mathbb{C}$  is an  $S(V)$ -module via the projection  $S(V) \rightarrow S^0(V) = \mathbb{C}$ ) is given by:

$$\sigma_u \cup \sigma_v = \sum_{\substack{w \in W: \\ \ell(w) = \ell(u) + \ell(v)}} p_{u,v}^w \sigma_w .$$

Note that the Littlewood-Richardson coefficients in (1.1) are given by

$$c_{u,v}^w = \delta_{\ell(w), \ell(u) + \ell(v)} p_{u,v}^w$$

for all  $u, v, w$ . In order to compute  $p_{u,v}^w$  we employ the relative Littlewood-Richardson coefficients  $p_{\mathbf{i}', \mathbf{i}''}^{\mathbf{i}}$  for  $\mathbf{i} \in R(w)$ ,  $\mathbf{i}' \in R(u)$ ,  $\mathbf{i}'' \in R(v)$  and use (4.9). That is, in view of Proposition 4.13, our Theorem 2.4 follows from Theorem 2.13 that we prove in the next section.

**Proof of Theorem 2.14.** Indeed, let  $\widehat{\mathcal{A}}_A$  be as in Proposition 4.16(b). That is,  $\mathcal{A}_A$  is dual (over  $S(V)$ ) to the generalized nil Hecke algebra  $\mathcal{H}_A(\widehat{W})$ , where  $\widehat{W}$  is the free monoid generated by  $S = \{s_i, i \in I\}$ . Since  $\widehat{\mathcal{A}}_A = \mathcal{A}(G)$  whenever  $A$  is the Cartan matrix of  $G$ , this proves Theorem 2.14(a).

Furthermore, let  $\mathcal{A}_A^{adm}$  be the dual (over  $S(V)$ ) to the generalized nil Hecke algebra  $\mathcal{H}_A(\widetilde{W})$ , where  $\widetilde{W}$  is the free Coxeter group generated by  $S = \{s_i, i \in I\}$ . Clearly, the canonical projection  $\widehat{W} \rightarrow \widetilde{W}$  extends to a surjective homomorphism of algebras

$$\mathcal{H}_A(\widehat{W}) \twoheadrightarrow \mathcal{H}_A(\widetilde{W})$$

commuting with the co-product. Dualizing, we see that  $\mathcal{A}_A^{adm}$  is a subalgebra of  $\widehat{\mathcal{A}}_A$  spanned (over  $S(V)$ ) by all  $\sigma_{\mathbf{i}}$ , where  $\mathbf{i}$  runs over all admissible sequences. This proves Theorem 2.14(b).

Furthermore, since  $\mathcal{A}_A(W) = H^*(G/B, \mathbb{C})$  whenever  $A$  is the Cartan matrix and  $W$  is the Weyl group of  $G$ , Proposition 4.16(c) proves Theorem 2.14(c).

Prove parts (d) and (e) of Theorem 2.14 now. For each  $\mathbf{i} \in I^m$  denote by  $\widehat{\mathcal{A}}_{\mathbf{i}, A}$  the algebra  $\mathcal{A}_{X, \mathbf{i}, S(V)}$  from Proposition 3.14(b), which, by definition is the dual of the subcoalgebra  $\mathcal{C}_{\mathbf{i}} \subset \mathcal{H}_A(\widehat{W})$  spanned by all  $x_{\mathbf{j}}$ , where  $\mathbf{j}$  runs over all subsequences of  $\mathbf{i}$ . Thus,  $\widehat{\mathcal{A}}_{A, \mathbf{i}}$  is quotient algebra  $\widehat{\mathcal{A}}_A / J_{\mathbf{i}}$ , where  $J_{\mathbf{i}}$  is the ideal spanned (over  $S(V)$ ) by all  $\sigma_{\mathbf{i}'}$  such that  $\mathbf{i}'$  is not a subsequence of  $\mathbf{i}$ .

Furthermore, in the notation of Theorem 3.18, for each sequence  $\mathbf{i} \in I^m$ , denote by  $\mathcal{BS}_{A, \mathbf{i}}$  the Bott-Samelson algebra  $\mathcal{BS}_{X, \mathbf{i}, S(V)}$ .

That is, taking into account that  $x_i(\alpha_j) = -a_{ij}$ ,  $\mathcal{BS}_{A, \mathbf{i}}$  is an  $S(V)$ -algebra generated by  $\sigma_1, \dots, \sigma_m$  subject to the relations:

$$(4.11) \quad \sigma_k^2 = \alpha_{i_k} \sigma_k - \sum_{\ell > k} a_{i_k, i_\ell} \sigma_\ell \sigma_k$$

for  $k \in [m]$ .

The following is a generalization of Proposition 2.9 to any Coxeter semigroup  $W = \langle s_i, i \in I \rangle$  and compatible quasi-Cartan matrix  $A$ .

**Corollary 4.20.** *(of Theorem 3.18) Let  $A$  be a quasi-Cartan matrix. Then for any sequence  $\mathbf{i} = (i_1, \dots, i_m) \in I^m$  the association*

$$(4.12) \quad \sigma_{\mathbf{j}}^{(\mathbf{i})} \mapsto \sum_{\substack{K \subset [m]: \\ \mathbf{i}_K = \mathbf{j}}} \sigma_K$$

*defines an injective homomorphism of  $R$ -algebras  $\pi_{\mathbf{i}} : \widehat{\mathcal{A}}_{A, \mathbf{i}} \hookrightarrow \mathcal{BS}_{A, \mathbf{i}}$ .*



Furthermore, if  $A$  is the Cartan matrix of  $G$ , then [56, Proposition 4] implies that

$$\mathcal{BS}_{A,\mathbf{i}} = H_T^*(\Gamma_{\mathbf{i}}(G), \mathbb{C}) .$$

Applying Corollary 4.20, we finish the proof of parts (d) and (e) of Theorem 2.14.

Theorem 2.14 is proved.  $\square$

## 5. PROOF OF THEOREMS 2.10, 2.4 AND 2.13

### 5.1. Proof of Theorem 2.10.

**Definition 5.1.** Let  $L, M \subset [m]$  such that  $|L| + |M| \geq m$ . We say that a map

$$\varphi : L \rightarrow \{0\} \cup [m] \setminus M$$

is *bounded* if:

- $\varphi(\ell) < \ell$  for each  $\ell \in L$ ;
- $|\varphi^{-1}(k)| = 1$  for all  $k \in [m] \setminus M$  (i.e., the restriction of  $\varphi$  to  $L' = \varphi^{-1}([m] \setminus M)$

is a bijection  $L' \xrightarrow{\sim} [m] \setminus M$ ).

Denote by  $V^\vee$  the  $\mathbb{k}$ -vector space with the basis  $\{\alpha_i^\vee, i \in I\}$  and define the pairing  $V \times V^\vee \rightarrow \mathbb{k}$  by  $\langle \alpha_i, \alpha_j^\vee \rangle = a_{ij}$  for  $i, j \in I$ . For each bounded map  $\varphi : L \xrightarrow{\sim} \{0\} \cup [m] \setminus M$  define  $p_\ell^{(\varphi)} \in V \sqcup \mathbb{k}$  by

$$(5.1) \quad p_\ell^{(\varphi)} = \begin{cases} \langle w_\ell^{(\varphi)}(\alpha_{i_\ell}), -\alpha_{i_{\varphi(\ell)}}^\vee \rangle & \text{if } \varphi(\ell) \neq 0 \\ w_\ell^{(\varphi)}(\alpha_{i_\ell}) & \text{if } \varphi(\ell) = 0 \end{cases}, \text{ where } w_\ell^{(\varphi)} = \prod_{\substack{r \in M \cup \varphi(L_{<\ell}) \\ \varphi(\ell) < r < \ell}} s_{i_r}$$

Clearly, there is a natural one-to-one correspondence between bounded maps  $\varphi : L \rightarrow \{0\} \cup [m] \setminus M$  and bounded bijections  $\varphi' : L' \xrightarrow{\sim} [m] \setminus M$ , where  $L'$  runs over all subsets of  $L$  such that  $|L'| + |M| = m$ . Therefore, Theorem 2.13 is equivalent to the following result.

**Proposition 5.2.** *For any repetition-free sequence  $\mathbf{i} = (i_1, \dots, i_m)$  and any subsequences  $\mathbf{i}', \mathbf{i}''$  of  $\mathbf{i}$  Theorem 5.3 holds. More precisely,*

$$(5.2) \quad p_{\mathbf{i}', \mathbf{i}''}^{\mathbf{i}} = \sum_{\varphi} \prod_{\ell \in L} p_\ell^{(\varphi)}$$

with the summation over all bounded maps  $\varphi : L \rightarrow \{0\} \cup [m] \setminus M$ , where  $L \subset M \subset [m]$  are determined by  $\{\mathbf{i}'\} \cap \{\mathbf{i}''\} = \{\mathbf{i}_L\}$ ,  $\{\mathbf{i}'\} \cup \{\mathbf{i}''\} = \{\mathbf{i}_M\}$  (in the notation of Proposition 3.15).

*Proof.* We prove Proposition 5.2 by induction in the length  $m$  of  $\mathbf{i}$ . If  $m = 0$ , i.e.,  $\mathbf{i} = \mathbf{i}' = \mathbf{i}'' = \emptyset$ ,  $p_{\mathbf{i}', \mathbf{i}''}^{\mathbf{i}} = 1$  and we have nothing to prove. Assume that  $m \geq 1$ . We apply the inductive hypothesis to  $\tilde{\mathbf{i}} = (i_2, \dots, i_m)$  and the subsequences  $\tilde{\mathbf{i}}' = \mathbf{i}' \setminus \{i_1\}$ ,  $\tilde{\mathbf{i}}'' = \mathbf{i}'' \setminus \{i_1\}$  of  $\tilde{\mathbf{i}}$ :

$$(5.3) \quad p_{\tilde{\mathbf{i}}', \tilde{\mathbf{i}}''}^{\tilde{\mathbf{i}}} = \sum_{\tilde{\varphi}} \prod_{\ell \in \tilde{L}} \tilde{p}_\ell^{(\tilde{\varphi})}$$

with the summation over all bounded maps  $\tilde{\varphi} : \tilde{L} \rightarrow \{0\} \cup \{2, \dots, m\} \setminus \tilde{M}$ , where  $\tilde{L} \subset \tilde{M} \subset \{2, \dots, m\}$  are determined by  $\{\tilde{\mathbf{i}}'\} \cap \{\tilde{\mathbf{i}}''\} = \{\tilde{\mathbf{i}}_L\}$ ,  $\{\tilde{\mathbf{i}}'\} \cup \{\tilde{\mathbf{i}}''\} = \{\tilde{\mathbf{i}}_{\tilde{M}}\}$  (in the notation of Proposition 3.15), and

$$(5.4) \quad \tilde{p}_\ell^{(\tilde{\varphi})} = \begin{cases} \langle \tilde{w}_\ell^{(\tilde{\varphi})}(\alpha_{i_\ell}), -\alpha_{i_{\tilde{\varphi}(\ell)}}^\vee \rangle & \text{if } \tilde{\varphi}(\ell) \neq 0 \\ \tilde{w}_\ell^{(\tilde{\varphi})}(\alpha_{i_\ell}) & \text{if } \tilde{\varphi}(\ell) = 0 \end{cases}, \text{ where } \tilde{w}_\ell^{(\tilde{\varphi})} = \prod_{\substack{r \in M \cup \tilde{\varphi}(L \setminus \tilde{L}) \\ \tilde{\varphi}(\ell) < r < \ell}} s_{i_r}$$

Since  $i_1$  is repetition-free, applying (3.16) to (5.3) (with  $k = 1$ ), we obtain:

$$(5.5) \quad p_{i', i''}^{\mathbf{i}} = \sum_{\tilde{\varphi}} f_{i_1} \left( \prod_{\ell \in \tilde{L}} \tilde{p}_\ell^{(\tilde{\varphi})} \right)$$

Consider three cases.

Case I.  $i'_1 = i''_1 = i_1$  so that  $f_{i_1} = \alpha_{i_1} s_{i_1}$  and  $L = \{1\} \cup \tilde{L}$ ,  $M = \{1\} \cup \tilde{M}$ . Clearly, each  $\tilde{\varphi} : L \setminus \{1\} \rightarrow \{0\} \cup [m] \setminus M$  as in (5.3) can be uniquely extended to a bounded map  $L \rightarrow \{0\} \cup [m] \setminus M$  by:  $\varphi(1) = 0$ . Thus,  $p_1^{(\varphi)} = \alpha_{i_1}$ ,  $p_\ell^{(\varphi)} = s_{i_1}(\tilde{p}_\ell^{(\tilde{\varphi})})$  for all  $\ell \in \tilde{L}$  and, therefore, (5.5) becomes (5.2). This proves (5.2) in Case I.

Case II.  $i'_1 \neq i''_1$ ,  $i_1 \in \{i'_1, i''_1\}$  so that  $f_{i_1} = s_{i_1}$  and  $L = \tilde{L}$ ,  $M = \{1\} \cup \tilde{M}$ . Therefore, each  $\tilde{\varphi}$  as in (5.3) is a bounded map  $L \rightarrow \{0\} \cup [m] \setminus M$ , i.e.,  $\tilde{\varphi} = \varphi$ . Thus,  $p_\ell^{(\varphi)} = s_{i_1}(\tilde{p}_\ell^{(\varphi)})$  for all  $\ell \in L$  and, therefore, (5.5) becomes (5.2). This proves (5.2) in Case II.

Case III.  $i_1 \notin \{i'_1, i''_1\}$  so that  $f_{i_1} = s_{i_1}$  and  $L = \tilde{L}$ ,  $M = \tilde{M}$ . Applying repeatedly the twisted Leibniz rule:

$$x_i(p_1 \cdots p_n) = x_i(p_1) s_i(p_2) \cdots s_i(p_n) + p_1 x_i(p_2) s_i(p_3) \cdots s_i(p_n) + \cdots p_1 \cdots p_{n-1} x_i(p_n)$$

for  $p_1, \dots, p_n \in S(V)$  and

$$x_i(\alpha) = \langle \alpha, -\alpha_i \rangle$$

for  $\alpha \in V$ , we obtain for each  $\tilde{\varphi} : L \rightarrow \{0\} \cup \{2, \dots, m\} \setminus M$  as in (5.3)

$$x_{i_1} \left( \prod_{\ell \in \tilde{L}} \tilde{p}_\ell^{(\tilde{\varphi})} \right) = \sum_{\substack{k \in L: \\ \tilde{\varphi}(k)=0}} \prod_{\ell \in L} p_\ell^{(\tilde{\varphi}, k)},$$

where for each  $k \in \tilde{\varphi}^{-1}(0)$ :

$$p_\ell^{(\tilde{\varphi}, k)} = \begin{cases} s_{i_1}(\tilde{p}_\ell^{(\tilde{\varphi})}) & \text{if } k < \ell \\ \langle \tilde{w}_\ell^{(\tilde{\varphi})}(\alpha_{i_k}), -\alpha_{i_1}^\vee \rangle & \text{if } k = \ell \\ \tilde{p}_\ell^{(\tilde{\varphi})} & \text{if } k > \ell \end{cases} = p_\ell^{(\varphi)},$$

where  $\varphi : L \rightarrow \{0\} \cup [m] \setminus M$  is a unique bounded map such that  $\varphi|_{L \setminus \{k\}} = \tilde{\varphi}|_{L \setminus \{k\}}$  and  $\varphi(k) = 1$ . By varying  $\tilde{\varphi}$ , we obtain all bounded maps  $L \rightarrow \{0\} \cup [m] \setminus M$ , i.e., (5.5) becomes (5.2). This proves (5.2) in Case III.

The proposition is proved.  $\square$

In view of (3.18) and (3.20), the assertion of Proposition 5.2 implies Theorem 2.10. Therefore, Theorem 2.10 is proved.  $\square$

**5.2. Proof of Theorems 2.4 and 2.13.** Since Theorem 2.4 directly follows from Theorem 4.19, Proposition 4.13, and Theorem 2.13, we will only prove the latter result in the following form.

**Theorem 5.3.** *For each triple of admissible sequences  $(\mathbf{i}, \mathbf{i}', \mathbf{i}'')$  such that  $\mathbf{i}'$  and  $\mathbf{i}''$  are sub-sequences of  $\mathbf{i}$  and the sum of lengths of  $\mathbf{i}'$  and  $\mathbf{i}''$  is greater or equal the length of  $\mathbf{i}$  one has:*

$$(5.6) \quad p_{\mathbf{i}', \mathbf{i}''}^{\mathbf{i}} = \sum \prod_{\ell \in K' \cap K''} p_{\ell}^{(\varphi)}$$

with the summation over all triples  $(K', K'', \varphi)$ , where

- $K', K'' \subset [m]$  such that  $\mathbf{i}_{K'} = \mathbf{i}'$ ,  $\mathbf{i}_{K''} = \mathbf{i}''$ ;
- $\varphi : K' \cap K'' \rightarrow \{0\} \cup [m] \setminus (K' \cup K'')$  is an  $\mathbf{i}$ -admissible bounded map.

The proof of the theorem will occupy the remainder of the section.

We now consider the general case where  $\mathbf{i}$  is not assumed to be repetition free.

We need the following result.

**Proposition 5.4.** *For each triple of admissible sequences  $(\mathbf{i}, \mathbf{i}', \mathbf{i}'')$  such that  $\mathbf{i}'$  and  $\mathbf{i}''$  are sub-sequences of  $\mathbf{i}$  and the sum of lengths of  $\mathbf{i}'$  and  $\mathbf{i}''$  is greater or equal the length of  $\mathbf{i}$  Theorem 5.3 holds if one drops the “ $\mathbf{i}$ -admissible” condition. More precisely, one has:*

$$(5.7) \quad p_{\mathbf{i}', \mathbf{i}''}^{\mathbf{i}} = \sum \prod_{\ell \in K' \cap K''} p_{\ell}^{(\varphi)}$$

with the summation over all triples  $(K', K'', \varphi)$ , where

- $K', K'' \subset [m]$  such that  $\mathbf{i}_{K'} = \mathbf{i}'$ ,  $\mathbf{i}_{K''} = \mathbf{i}''$ ;
- $\varphi : K' \cap K'' \rightarrow \{0\} \cup [m] \setminus (K' \cup K'')$  is a bounded map.

*Proof.* The assertion follows from (already proved) Theorem 2.10 and formula 3.19.  $\square$

Our next task is to show that equation (5.7) still holds if we restrict the sum to  $\mathbf{i}$ -admissible bounded maps. In order to prove we can make such a restriction, we need to develop some additional notation. For any subsets  $L \subseteq M$  of  $[m]$  such that  $|L| + |M| \geq m$  denote by  $P(L, M)$  the set of all bounded maps of  $L \rightarrow \{0\} \cup [m] \setminus M$  given by Definition 5.1. Let  $\mathbf{i} = (i_1, \dots, i_m) \in I^m$  be an admissible sequence (not necessarily repetition free) and let  $\mathbf{i}', \mathbf{i}''$  denote admissible subsequences of  $\mathbf{i}$  such that  $|\mathbf{i}'| + |\mathbf{i}''| \geq m$ . The following set will be important to the proceeding calculations.

Define  $J$  to be the set of all triples  $(K', K'', \varphi)$  which satisfy

- $K', K'' \subseteq [m]$  such that  $\mathbf{i}_{K'} = \mathbf{i}'$  and  $\mathbf{i}_{K''} = \mathbf{i}''$ .
- $\varphi \in P(K' \cap K'', K' \cup K'')$ .

Observe that the set  $J$  depends only on the data  $(\mathbf{i}', \mathbf{i}'', \mathbf{i})$ . We will use the capitol letter  $\Lambda := (\mathbf{i}', \mathbf{i}'', \varphi)$  to denote such triples. Proposition 5.4 is now equivalent to the equation

$$(5.8) \quad p_{\mathbf{i}', \mathbf{i}''}^{\mathbf{i}} = \sum_{\Lambda \in J} p_{\Lambda}$$

where if  $\Lambda = (K', K'', \varphi)$ , then  $p_\Lambda := \prod_{\ell \in K' \cap K''} p_\ell^{(\varphi)}$ .

Recall that for any sequence  $\mathbf{j} \in (I \times [m])^m$ , we say the bounded map  $\varphi$  is  $\mathbf{j}$ -admissible if the sequences  $\mathbf{j}_{M \cup (\varphi(L_{\leq \ell}) \setminus \{0\})}$  are admissible for all  $\ell \in L$ . For any sets  $L, M$  as above and sequence  $\mathbf{j} \in (I \times [m])^m$ , let  $P_{\mathbf{j}}(L, M) \subseteq P(L, M)$  denote the set of all  $\mathbf{j}$ -admissible bounded maps and let

$$J(\mathbf{j}) := \{(K', K'', \varphi) \in J \mid \varphi \in P_{\mathbf{j}}((K' \cap K'', K' \cup K'))\}.$$

Define the sequence

$$\mathbf{j}(k) := ((i_1, 1), (i_2, 1), \dots, (i_k, 1), (i_{k+1}, k+1), (i_{k+2}, k+2), \dots, (i_m, m))$$

and the set  $J(k) := J(\mathbf{j}(k))$ . It is easy to see that  $J(1) = J$  and that  $J(k) \subseteq J(k-1)$  for any  $k \in \{2, \dots, m\}$ . Theorem 5.3 is equivalent to showing that equation (5.8) still holds if we restrict the sum to  $J(m)$ . We will prove Theorem 5.3 by induction on  $k$ . Clearly for  $k = 1$ , we have that  $\mathbf{j}(1)$  repetition free and hence we are in the case of Proposition 5.4. It suffices to prove the following proposition.

**Proposition 5.5.** *With the assumptions in Theorem 5.3, we have that*

$$\sum_{\Lambda \in J(k-1) \setminus J(k)} p_\Lambda = 0$$

for any  $k \in \{2, \dots, m\}$ .

The remainder of this section consists of the proof for Proposition 5.5. Hence we will fix the integer  $k$  and denote the sequence  $\mathbf{j}(k)$  by simply  $\mathbf{j}$ . For  $\Lambda = (K', K'', \varphi) \in J(k-1) \setminus J(k)$ , define

$$L = K' \cap K'' = (\ell_1 < \dots < \ell_n) \quad \text{and} \quad M = K' \cup K'' = (m_1 < \dots < m_{n'}).$$

For any  $\ell_r \in L$  define

$$M(r) := M \cup (\varphi(L_{\leq \ell_r}) \setminus \{0\}).$$

For any subset  $N \subseteq [m]$ , we will denote by  $(N)$  the sequence of elements of  $N$  arranged in increasing order. We say that a pair  $\{n_1, n_2\} \subseteq N$  is non-admissible if  $\mathbf{j}_{n_1} = \mathbf{j}_{n_2}$  and  $n_1, n_2$  are consecutive in the sequence  $(N)$ . Since  $\mathbf{j}$  is fixed, this definition of non-admissible pair is well defined.

Since  $\Lambda = (K', K'', \varphi) \in J(k-1) \setminus J(k)$ , the bounded map  $\varphi$  is not  $\mathbf{j}$ -admissible. Hence, either  $\mathbf{j}_M$  is not admissible or  $\mathbf{j}_{M(r)}$  is not admissible for some  $\ell_r \in L$ . If  $\mathbf{j}_M$  is admissible, let  $z$  denote the smallest integer for which  $\mathbf{j}_{M(z)}$  is not admissible. We partition  $J(k-1) \setminus J(k)$  into the following sets:

$$\begin{aligned} J_1 &:= \{\Lambda \mid \mathbf{j}_M \text{ is not admissible}\}. \\ J_2 &:= \{\Lambda \mid \mathbf{j}_{M(z)} \text{ is not admissible and } \varphi(\ell_z), \ell_z \text{ are consecutive in } (M(z))\}. \\ J_3 &:= \{\Lambda \mid \mathbf{j}_{M(z)} \text{ is not admissible and } \varphi(\ell_z), \ell_z \text{ are not consecutive in } (M(z))\}. \end{aligned}$$

Observe that if  $\Lambda \in J_1$ , then  $M$  has a unique non-admissible pair since  $\Lambda \in J(k-1)$ . Similarly, if  $\Lambda \in J_2 \cup J_3$ , then  $M(z)$  has a unique non-admissible pair. We prove Proposition 5.5 in two steps.

**Proposition 5.6.** *The sum  $\sum_{\Lambda \in J_1 \cup J_2} p_\Lambda = 0$ .*

*Proof.* First suppose that  $\Lambda \in J_2$ . Then  $\{\varphi(\ell_z) < \ell_z\}$  is the unique non-admissible pair in  $M(z)$ . Define  $\Lambda_1 := (K'_1, K''_1, \varphi_1)$ ,  $\Lambda_2 := (K'_2, K''_2, \varphi_2)$  by

$$(K'_1, K''_1) := (K', K'' \ominus \{\varphi(\ell_z), \ell_z\}) \quad \text{and} \quad (K'_2, K''_2) := (K' \ominus \{\varphi(\ell_z), \ell_z\}, K'')$$

where  $\ominus$  denotes the symmetric difference operation. This implies that

$$L_1 = L_2 = L \setminus \{\ell_z\} \quad \text{and} \quad M_1 = M_2 = M \cup \{\varphi(\ell_z)\}$$

and hence we define

$$\varphi_1 = \varphi_2 = \varphi|_{L_1}.$$

Clearly we have  $\Lambda_1, \Lambda_2 \in J_1$  and that  $p_{\Lambda_1} = p_{\Lambda_2}$ . Moreover,

$$(5.9) \quad p_\Lambda + p_{\Lambda_1} + p_{\Lambda_2} = 0$$

since  $\langle \alpha_{\ell_z}, \alpha_{\varphi(\ell_z)} \rangle = 2$ .

Conversely, if  $\Lambda_1 = (K'_1, K''_1, \varphi_1) \in J_1$ , then let  $\{m_{z-1} < m_z\} \subseteq M$  denote the non-admissible pair in  $(M)$ . Note that  $\{m_{z-1}, m_z\} \cap L_1 = \emptyset$  since  $\mathbf{j}_{K'_1}$  and  $\mathbf{j}_{K''_1}$  are admissible. Without loss of generality, assume that  $m_z \in K'$  (hence  $m_{z-1} \in K''$ ) and define

$$\Lambda_2 := (K'_1 \ominus \{m_{z-1}, m_z\}, K''_1 \ominus \{m_{z-1}, m_z\}, \varphi_1)$$

and

$$\Lambda := (K'_1 \ominus \{m_{z-1}, m_z\}, K''_1, \varphi)$$

where  $\varphi = \varphi_1 \sqcup \{\varphi(m_z) = m_{z-1}\}$ . It is easy to see that  $\Lambda_2 \in J_1$ ,  $\Lambda \in J_2$ . and the triple  $(\Lambda_1, \Lambda_2, \Lambda)$  satisfies (5.9). Furthermore, the pairs  $\{\Lambda_1, \Lambda_2\}$  form an equivalence relation on  $J_1$  and the correspondence  $\{\Lambda_1, \Lambda_2\} \leftrightarrow \Lambda$  is a bijection between the set  $J_1$  modulo this equivalence relation and  $J_2$ . This proves the proposition.  $\square$

**Proposition 5.7.** *The sum  $\sum_{\Lambda \in J_3} p_\Lambda = 0$ .*

*Proof.* We prove the proposition by defining an involution on the set  $J_3$ . For any  $\Lambda \in J_3$ , define the set

$$\nu_\Lambda := \{\nu_1 < \nu_2\}$$

to be the non-admissible pair in the sequence  $(M(z))$ . The set  $\nu_\Lambda$  is well defined since the sequence  $\mathbf{j}_{M(z-1)}$  is admissible. Furthermore,  $\varphi(\ell_z) \in \nu_\Lambda$  and  $\ell_z \notin \nu_\Lambda$  since  $\Lambda \notin J_2$ .

For any subset  $N \subseteq [m]$  and  $\Lambda \in J_3$  define

$$\sigma_\Lambda(N) := \begin{cases} N \ominus \nu_\Lambda & \text{if } |N \cap \nu_\Lambda| = 1 \\ N & \text{if } |N \cap \nu_\Lambda| \neq 1 \end{cases}$$

where  $\ominus$  denotes the symmetric difference operation. If  $N = \{N_0\}$  is a set with a single element, then we will denote  $\sigma_\Lambda(N_0) := \sigma_\Lambda(\{N_0\})$  (dropping the brackets).

We define an involution  $\sigma : J_3 \rightarrow J_3$  by:

$$\sigma(\Lambda) := (\sigma_\Lambda(K'), \sigma_\Lambda(K''), \psi)$$

where  $\Lambda = (K', K'', \varphi)$  and  $\psi$  is defined as follows. It is easy to check that

$$\sigma_\Lambda(K') \cap \sigma_\Lambda(K'') = \sigma_\Lambda(L) \quad \text{and} \quad \sigma_\Lambda(K') \cup \sigma_\Lambda(K'') = \sigma_\Lambda(M).$$

Define  $\psi : \sigma_\Lambda(L) \rightarrow \{0\} \cup [m] \setminus \sigma_\Lambda(M)$  by

$$\psi(\sigma_\Lambda(\ell_k)) := \begin{cases} \sigma_\Lambda(\varphi(\ell_k)) & \text{if } \varphi(\ell_k) \neq 0 \\ 0 & \text{otherwise.} \end{cases}$$

The following properties are due to the fact that  $\nu_\Lambda$  is an admissible pair in  $M(\ell_z)$ , and  $\varphi(\ell_z) \in \nu_\Lambda$ . For any  $\Lambda$  and  $\sigma(\Lambda)$  we have

- $\sigma_\Lambda(\ell_r) = \ell_r$  for all  $r \geq z$
- $\psi(\sigma_\Lambda(\ell_r)) = \varphi(\ell_r)$  for all  $r > z$
- $\sigma_\Lambda(M)(r) = \sigma_\Lambda(M(r))$  for all  $r \in \{0, 1, \dots, n\}$ .

Clearly, by squaring we get  $\sigma^2(\Lambda) = \Lambda$  since  $\sigma_\Lambda^2(N) = N$  for any subset  $N \subseteq [m]$ . The following lemma proves that the image of  $\sigma$  is contained in  $J_3$  and hence  $\sigma$  is an involution.

**Lemma 5.8.** *If  $\Lambda \in J_3$ , then  $\sigma(\Lambda) \in J_3$ .*

*Proof.* Since  $\Lambda$  is fixed, we will denote  $\sigma_\Lambda(N)$  by simply  $\sigma(N)$  for any subset  $N$  in this proof. We first show that  $\sigma(\Lambda) \in J$ . Observe that  $\mathbf{i}_{\sigma(K')} = \mathbf{i}_{K'}$  and  $\mathbf{i}_{\sigma(K'')} = \mathbf{i}_{K''}$  since  $\mathbf{i}_{\nu_1} = \mathbf{i}_{\nu_2}$ . What we need to show is that  $\psi : \sigma(L) \rightarrow \{0\} \cup [m] \setminus \sigma(M)$  is a bounded map. It suffices to consider  $\ell_r$  for which  $\varphi(\ell_r) \neq 0$ .

If  $r > z$ , then we have that  $\sigma(M)(r) = M(r)$  since  $\nu_\Lambda \subseteq M(r)$ . Hence

$$\psi(\sigma(\ell_r)) = \varphi(\ell_r) < \ell_r = \sigma(\ell_r).$$

If  $r = z$ , then  $\varphi(\ell_r) \in \nu_\Lambda$ . But the fact that  $\{\sigma(\varphi(\ell_r)), \varphi(\ell_r)\} = \nu_\Lambda$  are consecutive in  $M(r)$  implies

$$\psi(\sigma(\ell_r)) = \sigma(\varphi(\ell_r)) < \ell_r = \sigma(\ell_r).$$

If  $r < z$ , then  $|M(r) \cap \nu_\Lambda| \leq 1$ . Hence  $\sigma$  fixes at least one of  $\ell_r$  or  $\varphi(\ell_r)$ . Thus

$$\psi(\sigma(\ell_r)) = \sigma(\varphi(\ell_r)) < \sigma(\ell_r)$$

since  $\nu_1, \nu_2$  are consecutive in  $M(z)$  and  $M(r) \subseteq M(z)$ . This implies that  $\psi$  is a bounded map. Hence  $\sigma(\Lambda) \in J$ .

Since  $\mathbf{j}_{\nu_1} = \mathbf{j}_{\nu_2}$ , we have that  $\mathbf{j}_{\sigma(M(k))} = \mathbf{j}_{M(k)}$  for all  $k \in \{0, 1, \dots, n\}$ . This implies that  $\Lambda \in J(k-1) \setminus J(k)$  if and only if  $\sigma(\Lambda) \in J(k-1) \setminus J(k)$ . In particular, it also implies that  $\Lambda \in J_3$  if and only if  $\sigma(\Lambda) \in J_3$ . This proves the lemma.  $\square$

Before we prove the proposition, we need one more observation. Note that each summand in equation (5.7) has a natural factorization

$$(5.10) \quad \prod_{\ell \in K' \cap K''} p_\ell^{(\varphi)} = \left( \prod_{\ell \in \varphi^{-1}(0)} p_\ell^{(\varphi)} \right) \left( \prod_{\ell' \in \varphi^{-1}([m] \setminus M)} p_{\ell'}^{(\varphi)} \right).$$

We will denote the first factor by  $p_\varphi^0$  and the second factor by  $p_\varphi^+$ .

**Lemma 5.9.** *For any  $\Lambda \in J_3$  with  $p_\Lambda = p_\Lambda^0 \cdot p_\Lambda^+$  and  $p_{\sigma(\Lambda)} = p_{\sigma(\Lambda)}^0 \cdot p_{\sigma(\Lambda)}^+$ , we have that  $p_\Lambda^0 = p_{\sigma(\Lambda)}^0$ .*

*Proof.* It suffices to check the case where  $\varphi^{-1}(0) \neq \sigma(\varphi^{-1}(0))$ . Otherwise  $p_\Lambda^0 = p_{\sigma(\Lambda)}^0$  since  $\nu_1$  and  $\nu_2$  act identically on  $Q$ .

If  $\varphi^{-1}(0) \neq \sigma(\varphi^{-1}(0))$ , then  $\{\varphi(\ell_z), \ell_r\}$  must be a non-admissible pair in  $M(z)$  for some  $r \neq z$ . Moreover,  $\varphi(\ell_r) = 0$  and  $r < z$ , otherwise  $\varphi$  would not be bounded. Since  $\{\varphi(\ell_z), \ell_r\}$  are a non-admissible pair in  $M(z)$ , they must be a non-admissible pair in  $M(r) \cup \{\varphi(\ell_z)\}$ . Thus

$$p_{\ell_r}^{(\varphi)} = p_{\sigma(\ell_r)}^{(\psi)}.$$

It is easy to see that other factors of  $p_\Lambda^0$  and  $p_{\sigma(\Lambda)}^0$  are equal. Thus the lemma is proved.  $\square$

We are now ready to prove the proposition. It suffices to show that  $p_\Lambda = -p_{\sigma(\Lambda)}$  for any  $\Lambda \in J_3$ . We first assume that  $\nu_2 = \varphi(\ell_z)$ . Then  $\nu_1 \in M(z-1)$  and hence

$$\nu_\Lambda \cap \sigma_\Lambda(M(z-1)) = \{\nu_2\} \quad \text{and} \quad \nu_\Lambda \subseteq \sigma_\Lambda(M(z)) = M(z).$$

For any  $i_\ell \in I$  we will denote  $\alpha_{i_\ell}$  by simply  $\alpha_\ell$ . By equation (2.2), if

$$p_\Lambda = p_\Lambda^0 \cdot p_\Lambda^+ = p_\Lambda^0 \cdot \prod_{\ell_r \in \varphi^{-1}([m] \setminus M)} \langle w_{\ell_r}^{(\varphi)}(-\alpha_{\ell_r}), \alpha_{\varphi(\ell_r)}^\vee \rangle,$$

then by Lemma 5.9, we have

$$\begin{aligned} p_{\sigma(\Lambda)} &= p_\Lambda^0 \cdot \langle s_{\nu_2} w_z^{(\psi)}(-\alpha_{\ell_z}), \alpha_{\varphi(\ell_z)}^\vee \rangle \prod_{r \neq z} \langle w_{\ell_r}^{(\psi)}(-\alpha_{\ell_r}), \alpha_{\varphi(\ell_r)}^\vee \rangle \\ &= p_\Lambda^0 \cdot \langle w_z^{(\varphi)}(\alpha_{\ell_z}), \alpha_{\varphi(\ell_z)}^\vee \rangle \prod_{r \neq z} \langle w_{\ell_r}^{(\varphi)}(-\alpha_{\ell_r}), \alpha_{\varphi(\ell_r)}^\vee \rangle \\ &= -p_\Lambda \end{aligned}$$

since  $\mathbf{j}_{\nu_1} = \mathbf{j}_{\nu_2}$ . Note that the other terms ( $r \neq z$ ) in the above product remain unchanged after applying the involution since  $\sigma_\Lambda(M(r)) = M(r)$  for  $r \geq z$  and if  $r < z$ , then the relative position of  $\nu_1$  and  $\nu_2$  is the same within the sequence  $(M(r))$ . A similar argument proves that  $p_\Lambda = -p_{\sigma(\Lambda)}$  in the case where  $\nu_1 = \varphi(\ell_z)$ . This completes the proof of Proposition 5.7  $\square$

Propositions 5.6 and 5.7 together prove Proposition 5.5. Hence the inductive step in the proof of Theorem 5.3 is complete.

## 6. POSITIVITY OF LITTLEWOOD-RICHARDSON COEFFICIENTS AND PROOF OF THEOREM 2.16

In this section we prove that the generalized Littlewood-Richardson coefficients are positive for a large class of quasi-Cartan matrices. The following is the main result of this section.

**Proposition 6.1.** *Let  $A$  be an  $I \times I$  quasi-Cartan matrix such that (2.19) holds. Then for any admissible sequence  $\mathbf{i} = (i_1, \dots, i_m) \in I^m$ , we have*

$$(6.1) \quad w(\alpha_j) \in \sum_{i \in I} \mathbb{R}_{\geq 0} \cdot \alpha_i \quad \text{and} \quad \langle w(\alpha_{i_m}), \alpha_{i_1}^\vee \rangle \leq 0$$

where  $w = s_{i_2} s_{i_3} \cdots s_{i_{m-1}}$ .

*Proof.* First, we consider the case where the quasi-Cartan matrix is of rank 2. Let  $I = \{1, 2\}$  and

$$A := \begin{bmatrix} 2 & -a \\ -b & 2 \end{bmatrix}.$$

Define the sequences  $A_k$  and  $B_k$  by

$$(6.2) \quad A_k := aB_{k-1} - A_{k-2} \quad \text{and} \quad B_k := bA_{k-1} - B_{k-2}$$

where  $A_0 = B_0 = 0$  and  $A_1 = B_1 = 1$ . These sequences are analogues of Chebyshev polynomials of second kind and are constructed so that if  $\mathbf{i} = (\underbrace{1, 2, 1, 2, \dots}_m)$ , then

$$w(\alpha_{i_m}) = A_{m-1} \alpha_1 + B_m \alpha_2 \quad \text{and} \quad \langle w(\alpha_{i_m}), \alpha_{i_1}^\vee \rangle = A_{m-2} - A_m$$

and if  $\mathbf{i} = (\underbrace{2, 1, 2, 1, \dots}_m)$ , then

$$w(\alpha_{i_m}) = A_m \alpha_1 + B_{m-1} \alpha_2 \quad \text{and} \quad \langle w(\alpha_{i_m}), \alpha_{i_1}^\vee \rangle = B_{m-2} - B_m.$$

We remark that the sequences  $A_k$  and  $B_k$  are used by N. Kitchloo in [20] in his study of cohomology of rank 2 Kac-Moody groups. The following lemma proves Proposition 6.1 (and hence Theorem 2.16) in the rank 2 case.

**Lemma 6.2.** *Let  $a, b$  be positive real numbers such that  $ab \geq 4$ , then for any admissible  $\mathbf{i} \in I^m$ , we have*

$$A_k \geq A_{k-2} \quad \text{and} \quad B_k \geq B_{k-2}.$$

*Proof.* We prove the lemma by induction on  $k$ . The lemma is clearly true for  $k = 2$  since  $a, b$  are positive. In general we have that

$$A_{k+1} = aB_k - A_{k-1} = (ab - 1)(A_{k-1}) - aB_{k-2} \geq 3A_{k-1} - aB_{k-2}.$$

By induction, we have that  $B_k \geq B_{k-2}$ . Hence

$$A_{k+1} \geq 3A_{k-1} - aB_{k-2} \geq 2A_{k-1} + (A_{k-1} - aB_k) = 2A_{k-1} - A_{k+1}.$$

This implies that  $2A_{k+1} \geq 2A_{k-1}$ . A similar argument proves the proposition for the sequence  $B_k$ . This completes the proof.  $\square$

We now consider the case of a quasi-Cartan matrix of arbitrary rank. For any  $j, k \in I$  let  $W_{j,k}$  denote the dihedral subgroup of  $W$  generated by  $s_j, s_k$ . We need the following well-known fact about Coxeter groups.



**Lemma 6.3.** *For any  $w \in W$  and  $j, k \in I$  there exist elements  $w' \in W$ ,  $w'' \in W_{j,k}$  such that*

$$(6.3) \quad w = w'w'', \quad \ell(w) = \ell(w') + \ell(w''), \quad \ell(w's_j) = \ell(w's_k) = \ell(w) + 1.$$

*In particular, the pair  $(w', w'')$  is unique and*

$$\ell(ws_j) - \ell(w) = \ell(w''s_j) - \ell(w''), \quad \ell(ws_k) - \ell(w) = \ell(w''s_k) - \ell(w'').$$

Now we prove Proposition 6.1 by induction in  $\ell(w)$ . If  $w \in W_{j,k}$  for some  $j, k \in I$ , then we are done by Lemma 6.2. Otherwise, by Lemma 6.3 there exists  $w' \in W \setminus \{1\}$  and  $w'' \in W_{j,k}$  satisfying (6.3). Since  $\ell(w'') < \ell(w)$  and  $w''$  satisfies the assumptions of the proposition, we obtain:

$$w''(\alpha_j) \in \mathbb{R}_{\geq 0} \cdot \alpha_j + \mathbb{R}_{\geq 0} \cdot \alpha_k$$

Since  $\ell(w') < \ell(w)$  and  $w'$  also satisfies the assumption of the proposition, the inductive hypothesis (6.1) applies to this  $w''$  and we obtain:

$$\begin{aligned} w(\alpha_j) &= w'w''(\alpha_j) \in w'(\mathbb{R}_{\geq 0} \cdot \alpha_j + \mathbb{R}_{\geq 0} \cdot \alpha_k) \\ &= \mathbb{R}_{\geq 0} \cdot w'(\alpha_j) + \mathbb{R}_{\geq 0} \cdot w'(\alpha_k) \subset \sum_{i \in I} \mathbb{R}_{\geq 0} \cdot \alpha_i. \end{aligned}$$

This proves the first part of (6.1). To prove the second part of (6.1), note that  $\ell(s_iws_j) - \ell(ws_j) = \ell(s_iw'') - \ell(w'') = 1$ . Therefore, the inductive hypothesis (6.1) applies to this  $w'$  and we obtain

$$\begin{aligned} \langle w(\alpha_j), \alpha_i^\vee \rangle &\in \langle \mathbb{R}_{\geq 0} \cdot w'(\alpha_j) + \mathbb{R}_{\geq 0} \cdot w'(\alpha_k), \alpha_i^\vee \rangle \\ &= \mathbb{R}_{\geq 0} \cdot \langle w'(\alpha_j), \alpha_i^\vee \rangle + \mathbb{R}_{\geq 0} \cdot \langle w'(\alpha_k), \alpha_i^\vee \rangle \subset \mathbb{R}_{\geq 0} \cdot \mathbb{R}_{\leq 0} + \mathbb{R}_{\geq 0} \cdot \mathbb{R}_{\leq 0} = \mathbb{R}_{\leq 0}. \end{aligned}$$

The proposition is proved.  $\square$

By replacing the quasi-Cartan matrix  $A$  with  $(1+t)A - 2t \cdot Id$ , we obtain the following result.

**Proposition 6.4.** *In the notation of Proposition 6.1, let  $A_t = (1+t) \cdot A - 2t \cdot Id$  be the  $I \times I$  quasi-Cartan matrix over  $\mathbb{R}[t]$ , where  $A$  is a quasi-Cartan matrix over  $\mathbb{R}$  such that for each  $i \neq j$  we have  $a_{ij} \leq 0$  and  $a_{ij}a_{ji} \geq 4$ . Then for any admissible sequence  $\mathbf{i} = (i_1, \dots, i_m) \in I^m$ , we have*

$$w(\alpha_{i_m}) \in \sum_{i \in I} \mathbb{R}_{\geq 0}[t] \cdot \alpha_i \quad \text{and} \quad \langle w(\alpha_{i_m}), \alpha_{i_1}^\vee \rangle \in \mathbb{R}_{\leq 0}[t].$$

where  $w = s_{i_2} \cdots s_{i_{m-1}}$ .

*Proof.* Define a partial order on  $\mathbb{R}[t]$  by saying that  $p \geq q$  if  $p - q \in \mathbb{R}_{\geq 0}[t]$ . Following the proof of Lemma 6.2 we obtain (by replacing  $(a, b)$  with  $((t+1)a, (t+1)b)$  and sequences  $\{A_k\}$ ,  $\{B_k\}$  with  $\{A_k(t)\}$ ,  $\{B_k(t)\} \subset \mathbb{R}[t]$ ). We prove by induction the following two statements

$$A_k(t) \geq A_{k-2}(t) \quad \text{and} \quad B_k(t) \geq B_{k-2}(t).$$

The lemma is clearly true for  $k = 2$  since  $A_2(t) = at + a$ ,  $B_2(t) = bt + b$  and  $a, b$  are positive. In general we have that

$$\begin{aligned} A_{k+1}(t) &= (at + a)B_k(t) - A_{k-1}(t) \\ &= (ab(t^2 + 2t) + ab - 1)A_{k-1}(t) - (at + a)B_{k-2}(t) \\ &\geq (ab(t^2 + 2t) + 3)A_{k-1}(t) - (at + a)B_{k-2}(t). \end{aligned}$$

By induction, we have that  $B_k(t) \geq B_{k-2}(t)$ . Hence

$$\begin{aligned} A_{k+1}(t) &\geq (ab(t^2 + 2t) + 3)A_{k-1}(t) - (at + a)B_{k-2}(t) \\ &\geq (ab(t^2 + 2t) + 2)A_{k-1}(t) + (A_{k-1}(t) - (at + a)B_k(t)) \\ &= (ab(t^2 + 2t) + 2)A_{k-1}(t) - A_{k+1}(t). \end{aligned}$$

This implies that

$$2(A_{k+1}(t) - A_{k-1}(t)) \geq ab(t^2 + 2t)A_{k-1}.$$

Similarly the polynomials  $B_k(t)$  satisfies the same inequality. This proves the proposition.  $\square$

Now we are ready to prove Theorem 2.16 and verify Conjecture 2.18 in a number of cases. Indeed, for any  $(K', K'', L, \varphi)$  as in Theorem 2.13, the sequence  $\mathbf{i}_{(K' \cup K'')_{<\ell} \cap \varphi(L_{<\ell})}$  is admissible for all  $\ell \in K' \cap K''$ , therefore,  $w_\ell(\alpha_{i_\ell}) \in \sum_{i \in I} \mathbb{R}_{\geq 0} \cdot \alpha_i$  for all  $\ell \in (K' \cap K'') \setminus L$  by (6.1) and  $\langle w_\ell(\alpha_{i_\ell}), -\alpha_{\varphi_{i_\ell}} \rangle \geq 0$  for all  $\ell \in L$ , again, by (6.1). This proves Theorem 2.16.  $\square$

Same argument, in conjunction with Proposition 6.4 verifies Conjecture 2.18 in the assumption that 2.19 holds.

## 7. EXAMPLES

In this section we apply Theorem 2.4 to compute Littlewood-Richardson coefficients in several cases. In the first example, we consider any rank 2 quasi-Cartan matrices and demonstrate that Theorem 2.4 agrees with formulas developed in [20] and [3]. The following examples we look at particular computations in finite Coxeter types  $A_n$  and  $H_3$ . The computer algebra program MuPAD Pro and 'Combinat' package was used in many of these calculations.

**7.1. The rank 2 case.** We give a full analysis in the case where  $A$  is a rank 2 quasi-Cartan matrix. Let  $I = \{1, 2\}$  and consider the quasi-Cartan matrix

$$A := \begin{bmatrix} 2 & -a \\ -b & 2 \end{bmatrix}$$

as in the previous section. Define

$$u_m = \underbrace{\cdots s_1 s_2 s_1}_m \quad \text{and} \quad v_m = \underbrace{\cdots s_2 s_1 s_2}_m$$

to be the unique elements in  $W$  corresponding to the two admissible sequences of length  $m$ . We first compute non-equivariant coefficients  $c_{u,v}^w$  in the case where  $\ell(u) + \ell(v) = \ell(w)$ . Let  $k \leq m$ . Theorem 2.4 implies that

$$\begin{aligned} c_{u_k, u_{m-k}}^{u_m} &= c_{v_{k+1}, u_{m-k}}^{v_{m+1}} = c_{u_k, v_{m-k+1}}^{v_{m+1}}, \\ c_{v_k, v_{m-k}}^{v_m} &= c_{u_{k+1}, v_{m-k}}^{u_{m+1}} = c_{v_k, u_{m-k+1}}^{u_{m+1}} \end{aligned}$$

and

$$c_{u_k, u_{m-k}}^{v_m} = c_{v_k, v_{m-k}}^{u_m} = 0.$$

Hence it suffices to compute coefficients  $c_{u_k, u_{m-k}}^{u_m}$  and  $c_{v_k, v_{m-k}}^{v_m}$ . Recall the sequences  $A_k$  and  $B_k$  defined in (6.2). For  $k \leq m$ , define the binomial coefficients

$$\begin{aligned} C(k, m) &:= \frac{A_m A_{m-1} \cdots A_1}{(A_k A_{k-1} \cdots A_1)(A_{m-k} A_{m-k-1} \cdots A_1)} \\ D(k, m) &:= \frac{B_m B_{m-1} \cdots B_1}{(B_k B_{k-1} \cdots B_1)(B_{m-k} B_{m-k-1} \cdots B_1)}. \end{aligned}$$

**Theorem 7.1.** *Let  $A$  be a rank 2 quasi-Cartan matrix. The coefficients*

$$c_{u_k, u_{m-k}}^{u_m} = C(k, m) \quad \text{and} \quad c_{v_k, v_{m-k}}^{v_m} = D(k, m).$$

We remark that the above formula has been proved by Kitchloo in [20, Section 10] in the case where  $A$  is the Cartan matrix of some Kac-Moody group and by the first author and Kapovich in [3, Section 13] in the case where  $A$  is symmetric. We show that Theorem 2.4 implies Theorem 7.1 for any rank 2 quasi-Cartan matrix. First, it is easy to check that Theorem 7.1 is true for  $m = 1$  and 2. We will show that the coefficients  $c_{u_k, u_{m-k}}^{u_m}$  and  $c_{v_k, v_{m-k}}^{v_m}$  can be constructed by a second order recurrence relation using Theorem 2.4. We will then show that  $C(k, m)$  and  $D(k, m)$  also satisfy this relation.

Let  $\mathbf{i} = (\dots, 1, 2, 1)$  be the reduced expression of  $u_m$ . If  $\mathbf{u}, \mathbf{v} \subset [m]$  are such that  $\mathbf{i}_{\mathbf{u}}$  and  $\mathbf{i}_{\mathbf{v}}$  are reduced expressions for  $u_m$  and  $u_{m-k}$  respectively, then there is at most one admissible bounded bijection

$$\varphi : \mathbf{u} \cap \mathbf{v} \rightarrow [m] \setminus (\mathbf{u} \cup \mathbf{v}).$$

Moreover, if  $\varphi$  exists, then  $[m] \setminus (\mathbf{u} \cup \mathbf{v}) = (1, 2, \dots, |\mathbf{u} \cap \mathbf{v}|)$ . Define

$$\mathcal{J}(m, k) := \{(\mathbf{u}, \mathbf{v}) \mid (\mathbf{i}_{\mathbf{u}}, \mathbf{i}_{\mathbf{v}}) \in R(u_m) \times R(u_{m-k}) \text{ and } \varphi \text{ exists}\}.$$

If  $\mathbf{u} \cap \mathbf{v} = \emptyset$ , our convention will be that  $\varphi$  exists. If  $\mathbf{z} \in \mathcal{J}(m, k)$ , then let  $\varphi_{\mathbf{z}}$  denote the corresponding  $\mathbf{i}$ -admissible bounded bijection. Theorem 2.4 says that

$$c_{u_k, u_{m-k}}^{u_m} = \sum_{\mathbf{z} \in \mathcal{J}(m, k)} p_{\varphi_{\mathbf{z}}}.$$

Define the subset

$$\mathcal{J}_1 := \{(\mathbf{u}, \mathbf{v}) \in \mathcal{J}(m, k) \mid m \in \mathbf{u} \cap \mathbf{v}\}.$$

If  $\mathbf{z} \in \mathcal{J}_1$ , then  $\varphi_{\mathbf{z}}(m) = 1$  since  $\varphi_{\mathbf{z}}$  is  $\mathbf{i}$ -admissible. Hence the partition  $\mathcal{J}(m, k) = \mathcal{J}_1 \sqcup \mathcal{J}(m, k) \setminus \mathcal{J}_1$  induces the recursion

$$\begin{aligned}
c_{u_k, u_{m-k}}^{u_m} &= \sum_{\mathbf{z} \in J_1} p_{\varphi_{\mathbf{z}}} + \sum_{\mathbf{z}' \in \mathcal{J}(m, k) \setminus J_1} p_{\varphi_{\mathbf{z}'}} \\
&= \langle v_{m-2}(-\alpha_1), \alpha_{i_1}^\vee \rangle \cdot c_{v_{k-1}, v_{m-k-1}}^{v_{m-2}} + (c_{u_{k-2}, u_{m-k}}^{u_{m-2}} + c_{u_k, u_{m-k-2}}^{u_{m-2}}).
\end{aligned}$$

Now assume that Theorem 7.1 is true for all integers less than  $m$ . Then

$$(7.1) \quad c_{u_k, u_{m-k}}^{u_m} = \langle w(-\alpha_{i_m}), \alpha_{i_1}^\vee \rangle D(k-1, m-2) + C(k-2, m-2) + C(k, m-2).$$

The following lemma will be important to the proceeding calculations.

**Lemma 7.2.** *Let  $A_m$  and  $B_m$  be sequence defined in (6.2). Then the following identities are true:*

- (1) *If  $m$  is odd, then  $A_m = B_m$ . If  $m$  is even, then  $bA_m = aB_m$ .*
- (2) *For any  $k \leq m$ , if  $k$  is odd and  $m$  is even, then*

$$bC(k, m) = aD(k, m).$$

*Otherwise*

$$C(k, m) = D(k, m).$$

- (3) *For any  $k \leq m$ , if  $k$  and  $m$  are both even, then*

$$bA_k A_m = a(A_{m+k-1} + A_{m+k-3} + \cdots + A_{m-k+1}).$$

*Otherwise*

$$A_k A_m = A_{m+k-1} + A_{m+k-3} + \cdots + A_{m-k+1}.$$

*Proof.* Part (1) follows from a simple inductive argument and the construction of  $A_m$  and  $B_m$  in (6.2). Part (2) is a direct consequence of part (1). For part (3) we observe that for any  $1 < k \leq m$ , we have

$$A_2 B_m = A_{m+1} + A_{m-1}$$

and

$$A_k B_m = A_2 A_k A_{m-1} - A_k B_{m-2}.$$

Part (3) now follows from another inductive argument and part (1).  $\square$

We prove Theorem 7.1 by considering three cases. First assume that  $m$  is odd. By Lemma 7.2, equation (7.1) becomes

$$\begin{aligned}
c_{u_k, u_{m-k}}^{u_m} &= C(m-2, k) + (A_m - A_{m-2})C(m-2, k-1) + C(m-2, k-2) \\
(7.2) \quad &= \tilde{A} \left( \frac{A_{m-2} A_{m-3} \cdots A_{m-k+1}}{A_k A_{k-1} \cdots A_1} \right)
\end{aligned}$$

where

$$\tilde{A} = A_{m-k} A_{m-k-1} + (A_m - A_{m-2}) A_k A_{m-k} + A_k A_{k-1}.$$

Using Lemma 7.2 part (3),  $\tilde{A}$  simplifies to

$$\tilde{A} = A_m A_{m-1}$$

and thus  $c_{u_k, u_{m-k}}^{u_m} = C(m, k)$ .

If  $k$  and  $m$  are both even, then equation (7.2) for  $c_{u_k, u_{m-k}}^{u_m}$  still holds by replacing  $\tilde{A}$  with

$$\begin{aligned}\tilde{A}' &= A_{m-k}A_{m-k-1} + (B_m - B_{m-2})A_kA_{m-k} + A_kA_{k-1} \\ &= A_{m-k}A_{m-k-1} + \frac{b}{a}(A_m - A_{m-2})A_kA_{m-k} + A_kA_{k-1}.\end{aligned}$$

But this expression still simplifies to equal  $A_mA_{m-1}$  by applying Lemma 7.2 part (3) in the case where  $k$  and  $m-k$  are both even.

Finally, if  $k$  is odd and  $m$  is even, then

$$\begin{aligned}c_{u_k, u_{m-k}}^{u_m} &= C(m-2, k) + (B_m - B_{m-2})D(m-2, k-1) + C(m-2, k-2) \\ &= C(m-2, k) + \frac{b}{a}(A_m - A_{m-2})\frac{a}{b}C(m-2, k-1) + C(m-2, k-2) \\ &= \tilde{A} \left( \frac{A_{m-2}A_{m-3} \cdots A_{m-k+1}}{A_kA_{k-1} \cdots A_1} \right)\end{aligned}$$

with  $\tilde{A}$  again simplifying to equal  $A_mA_{m-1}$ . To complete the proof of Theorem 7.1 we observe that this same argument applies to computing  $c_{v_k, v_{m-k}}^{v_m}$ .

**Remark 7.3.** In [3, Section 13], the first author and Kapovich consider the case where  $a = b = t + t^{-1}$  where  $t$  is some formal parameter. In this case

$$A_k = B_k = [k]_t := t^{k-1} + t^{k-3} + \cdots + t^{1-k}$$

and

$$C(k, m) = D(k, m) = \left[ \begin{matrix} m \\ k \end{matrix} \right]_t := \frac{[m]_t!}{[k]_t![m-k]_t!}$$

are  $t$ -binomial coefficients used in the study of quantum groups. Theorem 2.4 provides an interesting decomposition identity for these binomial coefficients.

We conclude our rank 2 examples by computing some equivariant Littlewood-Richardson coefficients  $c_{u,v}^w$  where  $\ell(u) + \ell(v) > \ell(w)$ . Let  $w = u_5$ ,  $u = u_3$  and  $v = u_4$ . Let  $[5] = (1', 2', 3', 4', 5')$  denote the index sequence of  $\mathbf{i} = (1, 2, 1, 2, 1)$ . It is easy to see that  $u_3$  appears as a subsequence four times given by the subsequences

$$(1', 2', 3'), (1', 2', 5'), (1', 4', 5'), (3', 4', 5')$$

and  $u_4$  appears once as the subsequence  $(2', 3', 4', 5')$ . These subsequences yield the following quadruples  $(\mathbf{u}, \mathbf{v}, L, \varphi)$  as in Theorem 2.13. In the table below, we list the set  $L' := (\mathbf{u} \cap \mathbf{v}) \setminus L$ .

$\mathbf{u}$	$\mathbf{v}$	$L'$	$\varphi$	$p_\varphi$	$\alpha$
$(1', 2', 3')$	$(2', 3', 4', 5')$	$(2', 3')$	$L = \emptyset$	1	$u_1(\alpha_2) \cdot v_2(\alpha_1)$
$(1', 2', 5')$	$(2', 3', 4', 5')$	$(2', 5')$	$L = \emptyset$	1	$u_1(\alpha_2) \cdot v_4(\alpha_1)$
$(1', 4', 5')$	$(2', 3', 4', 5')$	$(4', 5')$	$L = \emptyset$	1	$u_3(\alpha_2) \cdot v_4(\alpha_1)$
$(3', 4', 5')$	$(2', 3', 4', 5')$	$(3', 4')$	$(5') \mapsto (1')$	$\langle -v_3(\alpha_1), \alpha_1^\vee \rangle$	$v_1(\alpha_1) \cdot u_2(\alpha_2)$
$(3', 4', 5')$	$(2', 3', 4', 5')$	$(3', 5')$	$(4') \mapsto (1')$	$\langle -u_2(\alpha_2), \alpha_1^\vee \rangle$	$v_1(\alpha_1) \cdot v_4(\alpha_1)$
$(3', 4', 5')$	$(2', 3', 4', 5')$	$(4', 5')$	$(3') \mapsto (1')$	$\langle -v_1(\alpha_1), \alpha_1^\vee \rangle$	$u_3(\alpha_2) \cdot v_4(\alpha_1)$

In this case, all bounded bijections are also  $\mathbf{i}$ -admissible bounded bijections. Summing these terms gives

$$\begin{aligned} c_{u,v}^w = & u_1(\alpha_2) \cdot v_2(\alpha_1) + u_1(\alpha_2) \cdot v_4(\alpha_1) + u_3(\alpha_2) \cdot v_4(\alpha_1) + (A_5 - A_3) v_1(\alpha_1) \cdot u_2(\alpha_2) \\ & + (A_4 - A_3) v_1(\alpha_1) \cdot v_4(\alpha_1) + (A_3 - 1) u_3(\alpha_2) \cdot v_4(\alpha_1). \end{aligned}$$

For another example, let  $w = u_5$ ,  $u = v = u_3$ . Again, let  $[5] = (1', 2', 3', 4', 5')$  denote the index sequence  $\mathbf{i} = (1, 2, 1, 2, 1)$ . Using the notation of Theorem 2.13 we have the following quadruples  $(\mathbf{u}, \mathbf{v}, L, \varphi)$  (with  $L' = (\mathbf{u} \cap \mathbf{v}) \setminus L$ ).

$\mathbf{u}$	$\mathbf{v}$	$L'$	$\varphi$	$p_\varphi$	$\alpha$
$(1', 2', 3')$	$(1', 4', 5')$	$(1')$	$L = \emptyset$	1	$\alpha_1$
$(1', 4', 5')$	$(1', 2', 3')$	$(1')$	$L = \emptyset$	1	$\alpha_1$
$(1', 2', 3')$	$(3', 4', 5')$	$(3')$	$L = \emptyset$	1	$v_2(\alpha_1)$
$(3', 4', 5')$	$(1', 2', 3')$	$(3')$	$L = \emptyset$	1	$v_2(\alpha_1)$
$(1', 2', 5')$	$(3', 4', 5')$	$(5')$	$L = \emptyset$	1	$v_4(\alpha_1)$
$(3', 4', 5')$	$(1', 2', 5')$	$(5')$	$L = \emptyset$	1	$v_4(\alpha_1)$
$(3', 4', 5')$	$(3', 4', 5')$	$(3')$	$(4', 5') \mapsto (2', 1')$	$\langle -u_1(\alpha_2), \alpha_2^\vee \rangle \cdot \langle -v_3(\alpha_1), \alpha_1^\vee \rangle$	$\alpha_1$
$(3', 4', 5')$	$(3', 4', 5')$	$(4')$	$(3', 5') \mapsto (2', 1')$	$\langle -\alpha_1, \alpha_2^\vee \rangle \cdot \langle -v_3(\alpha_1), \alpha_1^\vee \rangle$	$u_2(\alpha_2)$
$(3', 4', 5')$	$(3', 4', 5')$	$(5')$	$(3', 4') \mapsto (2', 1')$	$\langle -\alpha_1, \alpha_2^\vee \rangle \cdot \langle -u_2(\alpha_2), \alpha_1^\vee \rangle$	$v_4(\alpha_1)$

Summing these nine terms gives

$$c_{u,v}^w = 2(\alpha_1 + v_2(\alpha_1) + v_4(\alpha_1)) + (A_3 - 1)((A_5 - A_3)\alpha_1 + A_2 u_2(\alpha_2)) + A_2(A_4 - A_2)v_4(\alpha_1).$$

In this case, there would be 20 terms in the above sum if we did not make the “admissible” restriction.

**7.2. Finite Type A examples.** In this section we demonstrate some calculations in finite type  $A_n$ . Let  $I = \{1, 2, \dots, n\}$  and  $A = (a_{i,j})$  be matrix where

$$a_{i,i} = 2, \quad a_{i,i+1} = a_{i,i-1} = -1, \quad \text{and} \quad a_{i,j} = 0 \text{ if } |i - j| > 1.$$

In this case  $W$  is the symmetric group generated by order 2 simple reflections  $\{s_i \mid i \in I\}$  with Coxeter relations

$$(s_i s_{i+1})^3 = (s_i s_j)^2 = 1$$

where  $|i - j| > 1$ . Let  $\mathbf{i} = (3, 2, 1, 3, 2)$  and  $w = s_3 s_2 s_1 s_3 s_2$ . We compute  $c_{u,v}^w$  where

$$u = s_1 s_3 = s_3 s_1 \quad \text{and} \quad v = s_1 s_3 s_2 = s_3 s_1 s_2.$$

Let  $[5] = (1', 2', 3', 4', 5')$  denote the index sequence of the reduced sequence  $\mathbf{i}$ . By Theorem 2.4, we need to find all triples  $(\mathbf{u}, \mathbf{v}, \varphi)$  which satisfy the conditions given in (2.3). In this case, there are four triples given by the table below.

$\mathbf{u}$	$\mathbf{v}$	$\varphi$	$p_\varphi$
$(3', 4')$	$(1', 3', 5')$	$(3') \mapsto (2')$	$\langle -\alpha_1, \alpha_2^\vee \rangle = 1$
$(1', 3')$	$(3', 4', 5')$	$(3') \mapsto (2')$	$\langle -\alpha_1, \alpha_2^\vee \rangle = 1$
$(3', 4')$	$(3', 4', 5')$	$(3', 4') \mapsto (1', 2')$	$\langle -\alpha_1, \alpha_3^\vee \rangle \cdot \langle -s_1 \alpha_3, \alpha_3^\vee \rangle = 0 \cdot 1 = 0$
$(3', 4')$	$(3', 4', 5')$	$(3', 4') \mapsto (2', 1')$	$\langle -\alpha_1, \alpha_2^\vee \rangle \cdot \langle -s_2 s_1 \alpha_3, \alpha_3^\vee \rangle = 1 \cdot -1 = -1$

Summing the numbers  $p_\varphi$ , we get that

$$c_{u,v}^w = 1 + 1 + 0 - 1 = 1.$$

This example demonstrates that for some triples,  $(\mathbf{u}, \mathbf{v}, \varphi)$ , we can have  $p_\varphi < 0$  under the conditions given in Theorem 2.4. Hence nonnegativity is not immediately implied by Theorem 2.4 for finite type  $A$  coefficients. Observe that the decomposition sum in (2.3) for  $c_{u,v}^w$  depends strongly on the choice of the reduced word of  $w$ . Instead, if we choose reduced word  $\mathbf{i}' = (2, 3, 1, 2, 1) \in R(w)$ , then there is only one term in the decomposition sum (2.3) given by

$\mathbf{u}$	$\mathbf{v}$	$\varphi$	$p_\varphi$
$(2', 5')$	$(2', 3', 4')$	$(2') \mapsto (1')$	$\langle -\alpha_3, \alpha_2^\vee \rangle = 1$

Again we get that  $c_{u,v}^w = 1$ , however the decomposition is obviously simpler and trivially positive. To measure the complexity of these decompositions we compute the polynomials  $c_{u,v}^{\mathbf{i}}(t)$  for each  $\mathbf{i} \in R(w)$  (recall these polynomials are defined at the end of the introduction). We get

$\mathbf{i}$	$c_{u,v}^{\mathbf{i}}(t)$
$(2, 3, 1, 2, 1)$	$t + 1$
$(2, 1, 3, 2, 1)$	$t + 1$
$(2, 3, 2, 1, 2)$	$(t + 1)^2$
$(3, 2, 3, 1, 2)$	$(t + 1)^3$
$(3, 2, 1, 3, 2)$	$(t + 1)^3$

Observe that, in this example, the polynomial  $c_{u,v}^{\mathbf{i}}(t)$  is invariant under commuting relations in  $R(w)$ . Also, each polynomial has nonnegative coefficients and evaluation at  $t = 0$  recovers the corresponding Littlewood-Richardson coefficient.

For a larger example, let  $\mathbf{i} = (5, 2, 3, 4, 3, 1, 2, 1)$  and  $w = s_5 s_2 s_3 s_4 s_3 s_1 s_2 s_1$ . Let

$$u = s_4 s_2 \quad \text{and} \quad v = s_3 s_4 s_3 s_1 s_2 s_1.$$

In this case,  $u$  has two reduced words and  $v$  has 19 reduced words. Of these, there are only three triples  $(\mathbf{u}, \mathbf{v}, \varphi)$  which satisfy the conditions in (2.3). We get that

$$c_{u,v}^w = 0 + 1 + 1 = 2.$$

If we take the reduced word  $\mathbf{i}' = (5, 2, 4, 3, 2, 1, 2, 4) \in R(w)$ , then there are ten triples which yield

$$c_{u,v}^w = -1 + 0 + 0 + 0 + 0 + 0 + 0 + 1 + 1 + 1 = 2.$$

As in the previous example, the polynomials  $c_{u,v}^{\mathbf{i}}(t)$  are invariant in the commutativity classes in  $R(w)$ . Of the 64 reduced word decompositions of  $w$ , we get 5 distinct polynomials, which correspond to the 5 commutativity classes in  $R(w)$ . These polynomials, along with the size of each commutativity class, is listed below.

$[\mathbf{i}]$	$  \mathbf{i}  $	$c_{u,v}^{\mathbf{i}}(t)$
$[(5, 2, 3, 4, 3, 1, 2, 1)]$	14	$(t+1) \cdot (2t^2 + 4t + 1) \cdot (t^2 + 2t + 2)$
$[(5, 2, 4, 3, 4, 1, 2, 1)]$	30	$(3t^2 + 6t + 2) \cdot (t+1)^3$
$[(5, 4, 3, 2, 3, 4, 1, 2)]$	5	$2(t+1)^6$
$[(5, 2, 4, 3, 4, 2, 1, 2)]$	12	$(2t^4 + 8t^3 + 11t^2 + 6t + 2) \cdot (t+1)^3$
$[(5, 2, 3, 4, 3, 2, 1, 2)]$	3	$(t+1) \cdot (t^2 + 2t + 2) \cdot (2t^4 + 8t^3 + 10t^2 + 4t + 1)$

Once again, observe that the coefficients of  $c_{u,v}^{\mathbf{i}}(t)$  are nonnegative.

**7.3. Finite type  $H_3$  examples.** Let  $\rho := 2 \cos\left(\frac{\pi}{5}\right)$  and consider the quasi-Cartan matrix

$$A := \begin{bmatrix} 2 & -\rho & 0 \\ -\rho & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}.$$

The group  $W$  has three order 2 generators  $s_1, s_2, s_3$  which satisfy the following relations:

$$(s_1 s_2)^5 = (s_1 s_3)^2 = (s_2 s_3)^3 = 1.$$

The group  $W$  has 120 elements with the longest element having a Coxeter length of 15. The group  $W$  is referred to as the finite Coxeter group  $H_3$ . While  $H_3$  appears in the classification of finite irreducible Coxeter groups, it does not appear in the classification of finite root systems in Lie theory. Hence  $H_3$  is not a Weyl group of any Lie group or Kac-Moody group. We give all structure coefficients  $c_{u,v}^w$  where  $\ell(u) = \ell(v) = 2$  and  $\ell(w) = 4$  in the form of a multiplication table of dual elements  $\{\sigma_u \mid \ell(u) = 2\}$ . There are 5 elements of length 2 elements which can be represented by the following sequences:

$$(12), (21), (23), (32), (13)$$

and 9 elements of length 4 represented by the sequences:

$$(1212), (2121), (1321), (2312), (2123), (3212), (1323), (1213), (2123).$$

	$\sigma_{12}$	$\sigma_{21}$
$\sigma_{12}$	$\sigma_{1212} + \rho \sigma_{2312} + \rho^2 \sigma_{3212}$	
$\sigma_{21}$	$\rho (\sigma_{1212} + \sigma_{2121}) + \sigma_{1321} + \rho \sigma_{3212}$	$\sigma_{2121} + 2\rho \sigma_{1321}$
$\sigma_{13}$	$\rho (\sigma_{2123} + \sigma_{3212} + \sigma_{2312}) + \sigma_{1321} + \sigma_{1231}$	$\rho (\sigma_{1321} + \sigma_{1231}) + \rho \sigma_{2321}$
$\sigma_{23}$	$\sigma_{2312} + \sigma_{1323} + \rho \sigma_{2123}$	$\sigma_{2321} + \sigma_{2123} + \rho \sigma_{1231}$
$\sigma_{32}$	$\rho (\sigma_{2312} + \sigma_{3212})$	$\sigma_{2312} + \sigma_{3212} + \rho \sigma_{1321}$



	$\sigma_{13}$	$\sigma_{23}$	$\sigma_{32}$
$\sigma_{13}$	$\rho^2 \sigma_{1231} + \rho (\sigma_{2123} + \sigma_{2321})$		
$\sigma_{23}$	$\rho^2 \sigma_{2123} + \sigma_{1231}$	$\rho^2 \sigma_{2123}$	
$\sigma_{32}$	$\sigma_{2312} + \sigma_{2321} + \sigma_{1321}$	$\rho \sigma_{1323}$	$\rho \sigma_{2312}$

Clearly, the Littlewood-Richardson numbers computed above are not all integral, however they are nonnegative since  $\rho$  is positive. This evidence supports Conjecture 2.17 on nonnegativity given in the introduction. We end by giving an example of the polynomial  $c_{u,v}^i(t)$  for  $H_3$ . Let  $w = (1, 2, 1, 2, 3, 1, 2)$ ,  $u = (3, 1, 2, 3)$  and  $v = (1, 3, 2)$ . In this case we get that  $c_{u,v}^w = \rho^2$ . The set  $R(w)$  has 5 elements with 3 commutativity classes. The polynomials  $c_{u,v}^i(t)$  are given by the following table where  $\bar{\rho} := \rho - 1$  and

$$P := (\rho t + \rho)^2(\rho t + \bar{\rho})(t + \rho).$$

<b>i</b>	$c_{u,v}^i(t)$
(1,2,1,2,3,1,2)	$P \cdot ((\rho^2 + 1)(t^2 + 2t) + \rho)(\rho^2 t^2 + 2\rho^2 t + \bar{\rho})$
(1,2,1,2,1,3,2)	$P \cdot ((\rho^2 + 1)(t^2 + 2t) + \rho)(\rho^2 t^2 + 2\rho^2 t + \bar{\rho})$
(2,1,2,1,2,3,2)	$P \cdot (t + \rho)(\rho^2 t^2 + 2\rho^2 t + \bar{\rho})(\rho^2 t^4 + 4\rho^2 t^3 + \rho^5 t^2 + \rho^2 \bar{\rho}^3 t + 1)$
(2,1,2,1,3,2,3)	$P \cdot (t + \rho)((\rho^2 + 1)(t^2 + 2t) + \bar{\rho})$
(2,1,2,3,1,2,3)	$P \cdot (t + \rho)((\rho^2 + 1)(t^2 + 2t) + \bar{\rho})$

Since  $\rho$  and  $\bar{\rho}$  are both positive numbers, all the above polynomials have positive coefficients.

## REFERENCES

- [1] D. Anderson, W. Fulton, Equivariant Cohomology in Algebraic Geometry, book in progress.
- [2] P. Belkale, S. Kumar, Eigenvalue problem and a new product in cohomology of flag varieties, *Inventiones Math.* **166**, pp. 185–228 (2006).
- [3] A. Berenstein, M. Kapovich, Stability inequalities and universal Schubert calculus of rank 2, arXiv:1008.1773, 2010.
- [4] A. Berenstein, R. Sjamaar, *Projections of coadjoint orbits, moment polytopes, and the Hilbert-Mumford criterion*, *J. Amer. Math. Soc.* **13** (2000), 433–466.
- [5] N. Bergeron and F. Sottile, Schubert polynomials, the Bruhat order, and the geometry of flag manifolds, *Duke Math. J.* **95** (1998), no. 2, pp. 373–423.
- [6] N. Bergeron and F. Sottile, A Pieri-type formula for isotropic flag manifolds, *Trans. Amer. Math. Soc.* **354** (2002), pp. 4815–4829.
- [7] A. Bertram, I. Ciocan-Fontanine, W. Fulton, Quantum multiplication of Schur polynomials. *J. Algebra* **219** (1999), no. 2, pp. 728–746.
- [8] S. Billey, Kostant Polynomials and the cohomology ring for  $G/B$ , *Duke Math. J.* **96**(1) (1999), pp. 205–224.
- [9] S. Billey, M. Haiman, Schubert polynomials for the classical groups, *J. Amer. Math. Soc.* **8** (1995), pp. 443–482.
- [10] A. Buch, A. Kresch, H. Tamvakis, Littlewood-Richardson rules for Grassmannians, *Adv. Math.* **185** (2004), no. 1, 8090.
- [11] I. Coskun, A Littlewood-Richardson rule for two-step flag varieties, *Inventiones Mathematicae* vol. 176, no. **2** (2009) pp. 325–395.
- [12] H. Duan, Multiplicative rule of Schubert classes, *Invent. Math.*, 159 (2005), pp. 407–436.
- [13] S. Fomin, A. N. Kirillov, Combinatorial  $B_n$ -analogues of Schubert polynomials, *Trans. Amer. Math. Soc.* **348** (1996), pp. 3591–3620.

- [14] W. Fulton. *Young tableaux: With applications to representation theory and geometry*, volume 35 of *London Mathematical Society Student Texts*. Cambridge University Press, Cambridge, 1997.
- [15] W. Fulton, Eigenvalues, invariant factors, highest weights, and Schubert calculus, *Bull. Amer. Math. Soc.* **37** (2000), pp. 209-249.
- [16] M. Goresky, R. Kottwitz, R. MacPherson Equivariant Cohomology, Koszul Duality, and the Localization Theorem, *Invent. math.* **131** (1998), no. 1, pp. 25–83.
- [17] W. Graham, Positivity in equivariant Schubert calculus, *Duke Math. J.* **109** (2001), pp. 599–614.
- [18] F. Hivert, A. Schilling, N. Thiery, Hecke group algebras as quotients of affine Hecke algebras at level 0. *J. Combin. Theory Ser. A* **116** (2009), no. 4, pp. 844-863.
- [19] Anatol Kirillov, Skew divided difference operators and Schubert polynomials, *SIGMA (Symmetry Integrability Geom. Methods Appl.)* **3** (2007).
- [20] N. Kitchloo, On the topology of Kac-Moody groups, arXiv:0810.0851, 2008.
- [21] S. Kleiman, The transversality of a general translate, *Compositio Math.* **28** (1974), pp. 287–297.
- [22] A. Klyachko, Stable bundles, representation theory and Hermitian operators, *Selecta Math.* **3** (1998), pp. 419–445.
- [23] A. Knutson, Descent-cycling in Schubert calculus, *Experiment. Math.* **10** (2001), no. 3, pp. 345–353.
- [24] A. Knutson, A Schubert calculus recurrence from the noncomplex  $W$ -action on  $G/B$ , arXiv:math/0306304, (2003).
- [25] A. Knutson, K. Purbhoo, Product and puzzle formulae for  $GL_n$  Belkale-Kumar coefficients, arXiv:1008.4979, 2010.
- [26] A. Knutson, T. Tao. Puzzles and (equivariant) cohomology of Grassmannians, *Duke Math. J.* **119** (2003), pp. 221–260.
- [27] A. Knutson, T. Tao, C. Woodward. The honeycomb model of  $GL_n(\mathbb{C})$  tensor products II. Puzzles determine facets of the Littlewood-Richardson cone, *J. Amer. Math. Soc.* **1** (2004), pp. 19–48.
- [28] A. Knutson, A. Yong, A formula for  $K$ -theory truncation Schubert calculus. *Int. Math. Res. Not.* 2004, no. 70, pp. 3741-3756.
- [29] M. Kogan, RC-graphs and a generalized Littlewood-Richardson rule, *Internat. Math. Res. Notices* 2001, no. 15, pp. 765-782.
- [30] V. Kreiman, Equivariant Littlewood-Richardson skew tableaux, *Trans. Amer. Math. Soc.* **362** (2010), no. 5, pp. 2589-2617.
- [31] B. Kostant, S. Kumar, The nil Hecke ring and cohomology of  $G/P$  for a Kac-Moody group  $G$ , *Adv. in Math.*, 62(3) 1986, pp. 187–237.
- [32] M. Kogan, RC-graphs and a generalized Littlewood-Richardson rule, *Int. Math. Res. Not.* **15** (2001), pp. 765-782.
- [33] S. Kumar, The nil Hecke ring and singularity of Schubert varieties, *Inventiones Math.* **123**, (1996), pp. 471-506.
- [34] S. Kumar, Kac-Moody groups, their flag varieties and representation theory, *Progress in Mathematics*, vol. 204, Birkhäuser, Boston, MA, 2002.
- [35] S. Kumar and M. Nori, Positivity of the cup product in cohomology of flag varieties associated to Kac-Moody groups, *Internat. Math. Res. Notices*, **14** (1998), pp. 757–763.
- [36] A. Lascoux, M.P. Schützenberger, Schubert polynomials (French), *C.R. Acad. Sci. Paris Ser. I Math.* **294** (1982), pp. 447–450.
- [37] A. Lascoux, M.P. Schützenberger, Schubert polynomials and the Littlewood-Richardson rule, *Lett. Math. Phys.* **10** (1985), no. 2-3, 111-124.
- [38] L. Lapointe, J. Morse, Quantum cohomology and the  $k$ -Schur basis, *Trans. Amer. Math. Soc.* **360** (2008), no. 4, pp. 2021-2040.
- [39] C. Lenart, Growth diagrams for the Schubert multiplication, *Journal of Combinatorial Theory, Series A*, Vol. 117, No. 7 (2010), pp. 842-856.

- [40] I. Macdonald, Notes on Schubert Polynomials, *Publ. LaCIM* **6**, Université du Québec, Montréal, (1991).
- [41] L. Mihalcea, Equivariant quantum Schubert calculus, *Adv. Math.* **203** (2006), no. 1, 1-33.
- [42] A. Molev, Littlewood-Richardson problem for Schubert polynomials. *Austral. Math. Soc. Gaz.* **31** (2004), no. 5, pp. 295-297.
- [43] A. Molev, Littlewood-Richardson polynomials, *J. Algebra* **321** (2009), no. 11, pp. 3450-3468.
- [44] A. Postnikov, R. Stanley, Chains in the Bruhat order, *J. Algebraic Combin.* **29** (2009), no. 2, 133-174.
- [45] K. Purbhoo, F. Sottile, A Littlewood-Richardson rule for Grassmannian permutations, *Proc. Amer. Math. Soc.* **137** (2009), pp. 1875-1882.
- [46] N. Ressayre, Geometric invariant theory and the generalized eigenvalue problem, *Invent. Math.* **180** (2010), no. 2, pp. 389-441.
- [47] Richardson, R. W.; Springer, T. A. The Bruhat order on symmetric varieties, *Geom. Dedicata* **35** (1990), no. 1-3, pp. 389-436.
- [48] E. Richmond, A multiplicative formula for structure constants in the cohomology of flag varieties, *to appear in Michigan Math Journal* arXiv:0812.1856, 2008.
- [49] E. Richmond, A partial Horn recursion in the cohomology of flag varieties, *Journal of Algebraic Combinatorics*, Vol. 30 , **1** (2009) pp. 1 – 17.
- [50] J. Stembridge, On the full commutative elements of Coxeter groups, *Journal of Algebraic Combinatorics* **5** (1996), pp. 353-385.
- [51] H. Thomas and A. Yong, A combinatorial rule for (co)minuscule Schubert calculus, *Adv. Math.* **222** (2009), no. 2, pp. 596-620.
- [52] S. Tsaranov, Classification of Coxeter type monoids that have attractors, *Soviet Math. Dokl.* **39** (1989), no. 1, pp. 218-220.
- [53] S. Tsaranov, Representation and classification of Coxeter monoids, *European J. Combin.* **11** (1990), no. 2, pp. 189-204.
- [54] R. Vakil, A geometric Littlewood-Richardson rule (with an appendix joint with A. Knutson), *Annals of Math.* **164** (2006), pp. 371-422.
- [55] E.B. Vinberg, Discrete linear groups that are generated by reflections, *Izv. Akad. Nauk SSSR Ser. Mat.*, **35** (1971) pp. 1072-1112.
- [56] M. Willems, Equivariant cohomology and K-theory of Bott-Samelson varieties and flag varieties (French), *Bull. Soc. Math. France* **132** (2004), pp. 569-589.
- [57] M. Willems, Equivariant cohomology of Bott towers and equivariant Schubert calculus (French), *Journal of the Institute of Mathematics of Jussieu*, **5** (2006) pp. 125-159.

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